Statistical Digital Signal Processing

Week 8 Optimum Filters: Discrete Kalman Filter

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Previous Topic (Week-7)

Optimum Filters: IIR Wiener Filters

- Basics of IIR Wiener Filters
- Non-Causal IIR Wiener Filters

- The IIR Weiner Smoothing Filter
- Causal IIR Wiener Filters
- Causal Linear Prediction

Lecture Learning Outcomes

1. Explain the fundamental concepts, assumptions, and applications of the Kalman Filter in discrete-time dynamic systems
2. Derive and interpret the mathematical formulation of the Discrete Kalman Filter
3. Analyze the role of the Kalman Gain in minimizing estimation error and improving state estimation accuracy
4. Design a basic Discrete Kalman Filter for estimation and prediction of dynamic system states in the presence of process and measurement noise
5. Evaluate the performance of the Discrete Kalman Filter through system error covariance analysis.

Week 8: Discrete Kalman Filter

Outline

- Discrete Kalman Filter: Introduction
- Discrete Kalman Filter
- Kalman Gain, State Variable, and State Transition Matrix

Discrete Kalman Filter: Introduction

- So far, we have seen Wiener filter to recover the important signal $d(n)$ from its noisy observation, $x(n)$, considering $d(n)$ and $x(n)$ are jointly wide sense stationary

$$x(n) = d(n) + v(n) \qquad (1)$$

Where:

$v(n)$ Is noise signal

- However, most of the signal in practical system are not stationary
- This non-stationarity related constraint puts a limitation in the versatility and usefulness of Wiener filter
- Therefore, linear estimation should be analyzed in the context of **non stationarity** and **Discrete Kalman** Filter will step-in at this point

Discrete Kalman Filter

- Let's begin by example from previous lesson for estimating $x(n)$ by using causal Wiener filter from noisy observation $y(n)$:

$$y(n) = x(n) + v(n) \qquad (2)$$

Where: $v(n)$ Is a unit variance white noise uncorrelated with $x(n)$

- **Example:** Suppose $x(n)$ is an **AR(1)** process generated by the following difference equation:

$$x(n) = 0.8x(n-1) + w(n) \qquad (4)$$

Where: $w(n)$ is a white noise with variance $\sigma_w^2 = 0.36$ and

$$r_x(k) = 0.8^{|k|} \qquad (5)$$

- **Objective:** to find the **optimum causal IIR Winer** filter to estimate $x(n)$ from $y(n)$

Discrete Kalman Filter

- Recalling, :

$$P_{xy}(z) = P_x(z) \quad (6)$$

$$P_y(z) = P_x(z) + P_v(z) = P_x(z) + 1 \quad (7)$$

- $P_x(z)$ is also given by

$$P_x(z) = z\{r_x(k)\} = z\left\{0.8^{|k|}\right\}$$

$$= \frac{0.36}{\left(1-0.8z^{-1}\right)\left(1-0.8z\right)} \quad (8)$$

- The power spectrum of $y(n)$ becomes;

$$P_y(z) = P_x(z) + 1 = 1 + \frac{0.36}{\left(1-0.8z^{-1}\right)\left(1-0.8z\right)}$$

Discrete Kalman Filter

$$= 1.6 \frac{\left(1-0.5z^{-1}\right)\left(1-0.5z\right)}{\left(1-0.8z^{-1}\right)\left(1-0.8z\right)} \qquad (9)$$

- Since $P_y(z)$ is real, it can be factorized as:

$$P_y(z) = \sigma_0^2 Q(z)Q(z^{-1}) = 1.6\left[\frac{1-0.5z^{-1}}{1-0.8z^{-1}}\right]\left[\frac{1-0.5z}{1-0.8z}\right] \qquad (10)$$

- From eq(10), the following are obtained;

$$\sigma_0^2 = 1.6 \qquad Q(z) = \left[\frac{1-0.5z^{-1}}{1-0.8z^{-1}}\right] \qquad Q(z^{-1}) = \left[\frac{1-0.5z}{1-0.8z}\right] \qquad (11)$$

Discrete Kalman Filter

- Recalling the causal IIR Wiener filter is given by:

$$H(z)=\frac{1}{\sigma_0^2 Q(z)}\left[\frac{P_{xy}(z)}{Q(z^{-1})}\right]_+ \qquad (12)$$

- Then, we have ;

$$\frac{P_{xy}(z)}{Q(z^{-1})}=\frac{P_x(z)}{Q(z^{-1})}=\left[\frac{0.36}{\left(1-0.8z^{-1}\right)\left(1-0.8z\right)}\right]\left[\frac{1-0.8z}{1-0.5z}\right]$$

$$=\frac{0.36z^{-1}}{\left(1-0.8z^{-1}\right)\left(z^{-1}-0.5\right)}$$

$$=\frac{0.6}{1-0.8z^{-1}}+\frac{0.3}{z^{-1}-0.5} \qquad (13)$$

Discrete Kalman Filter

- Therefore:

$$\left[\frac{P_{xy}(z)}{Q(z^{-1})}\right]_+ = \frac{0.6}{1-0.8z^{-1}} \qquad (14)$$

- Now, we have ;

$$H(z) = \frac{1}{\sigma_0^2 Q(z)}\left[\frac{P_{xy}(z)}{Q(z^{-1})}\right]_+ = \frac{1}{1.6}\left[\frac{1-0.8z^{-1}}{1-0.5z^{-1}}\right]\left[\frac{0.6}{1-0.8z^{-1}}\right] = \frac{0.375}{1-0.5z^{-1}} \qquad (15)$$

- In time domain:

$$h(n) = 0.375\left(\frac{1}{2}\right)^n u(n) \qquad (16)$$

- Since $\hat{X}(z) = H(z)Y(z)$, the estimate of $x(n)$ can be found recursively:

$$\hat{x}(n) = 0.5\hat{x}(n-1) + 0.375y(n) \qquad (17)$$

Discrete Kalman Filter

- Interestingly, the previous equation, eq(17), can be written in the following alternative form:

$$\hat{x}(n) = 0.8\hat{x}(n-1) + 0.375\left[y(n) - 0.8\hat{x}(n-1)\right] \qquad (18)$$

- Given the observation $y(n)$ for estimating the desired signal $x(n)$

$$y(n) = x(n) + v(n) \qquad (19)$$

- Therefore, considering specific problem of estimating an $\mathrm{AR}(1)$ process of the form:

$$x(n) = a(1)x(n-1) + w(n) \qquad (20)$$

Where: $v(n)$ and $w(n)$ are uncorrelated white noise process

- What we have found out was, the estimated signal $\hat{x}(n)$ can

  be written in the form:

$$\hat{x}(n) = a(1)\hat{x}(n-1) + K\left[y(n) - a(1)\hat{x}(n-1)\right] \qquad (21)$$

Where: K is constant which is called **Kalman Gain**

Discrete Kalman Filter

- The **Kalman Gain,** K**,** minimizes the following mean square error:

$$\varepsilon_{\min} = E\left\{ \left| x(n) - \hat{x}(n) \right|^2 \right\} \qquad (22)$$

- However, the solution given in Eq (21) is optimum estimate of $x(n)$ if and only if $x(n)$ and $y(n)$ are jointly stationary
- If $x(n)$ nonstationary process, which can be generated by the following time varying process:

$$x(n) = a_{n-1}(1)x(n-1) + w(n) \qquad (23)$$

- The optimum estimate of $x(n)$ can be written as :

$$\hat{x}(n) = a_{n-1}(1)\hat{x}(n-1) + K(n)\left[y(n) - a_{n-1}(1)\hat{x}(n-1) \right] \qquad (24)$$

Where:

$K(n)$ is a suitable chosen **Gain**

Discrete Kalman Filter

- Although the discussion we made so far is focused on estimation of $\mathrm{AR}(1)$ process from noisy observations, we can extend the analysis to general case using **state variables**
- Lets consider $x(n)$ is $\mathrm{AR}(p)$ process generated by the following difference equation:

$$x(n) = \sum_{k=1}^{p} a(k)x(n-k) + w(n) \qquad (25)$$

- And $x(n)$ is measured in the presence of white noise:→ $y(n) = x(n) + v(n)$
- Defining $\mathbf{x}(n)$ is a $p-\mathrm{dimentional}$ state vector given by:

$$\mathbf{x}(n) = \begin{bmatrix} x(n) \\ x(n-1) \\ \vdots \\ x(n-p+1) \end{bmatrix} \quad \text{And} \quad \mathbf{x}(n-1) = \begin{bmatrix} x(n-1) \\ x(n-2) \\ \vdots \\ x(n-p) \end{bmatrix} \qquad (26)$$

Discrete Kalman Filter

- Writing Eq(25) in matrix form in terms of $\mathbf{x}(n)$ as:

$$\mathbf{x}(n) = \underbrace{\begin{bmatrix} a(1) & a(2) & \cdots & a(p-1) & a(p) \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix}}_{\mathbf{A}} \mathbf{x}(n-1) + \underbrace{\begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} w(n)}_{\mathbf{w}(n)} \qquad (27)$$

- The noisy observation → $y(n) = x(n) + v(n)$ can also be written as:

$$y(n) = [1,0,0,\ldots,0]\mathbf{x}(n) + v(n) \qquad (28)$$

Discrete Kalman Filter

- Eq(27) and Eq(28) can be written in compact matrix form as:

$$\mathbf{x}(n) = \mathbf{A}\mathbf{x}(n-1) + \mathbf{w}(n) \qquad (29)$$

$$y(n) = \mathbf{c}^T \mathbf{x}(n) + v(n) \qquad (30)$$

Where:

$\mathbf{A}$ is a $p \times p$ state transition matrix

$\mathbf{w}(n) = \left[w(n), 0, \ldots, 0\right]^T$ is a vector noise process

$\mathbf{c}$ is a unit vector of length p

- The optimum estimate of state vector $\mathbf{x}(n)$ can be given by:

$$\hat{\mathbf{x}}(n) = \mathbf{A}\hat{\mathbf{x}}(n-1) + \mathbf{K}\left[y(n) - \mathbf{c}^T \mathbf{A}\hat{\mathbf{x}}(n-1)\right] \qquad (31)$$

Where:

$\mathbf{K}$ is the **Kalman Gain Vector**

Discrete Kalman Filter

- Eq(31) is only applicable for stationary $\mathrm{AR}(p)$
- But it is possible to modify Eq(31) for generalized non stationary process
- Assuming $\mathbf{x}(n)$ the state vector with a dimension p which is given by:

$$\mathbf{x}(n) = \mathbf{A}(n-1)\mathbf{x}(n-1) + \mathbf{w}(n) \qquad (32)$$

Where:

$\mathbf{A}(n-1)$ Is at time varying $p \times p$ state transition matrix

$\mathbf{w}(n)$ Is zero mean white noise process

- From the definition of white noise, $w(n)$ satisfies:

$$E\left\{\mathbf{w}(n)\mathbf{w}^{H}(k)\right\} = \begin{cases} \mathbf{Q}_{w}(n) & ; \; k = n \\ 0 & ; \; k \neq n \end{cases} \qquad (33)$$

Discrete Kalman Filter

- Considering $\mathbf{y}(n)$ is also a vector of observation with length q formed by:

$$\mathbf{y}(n) = \mathbf{C}(n)\mathbf{x}(n) + \mathbf{v}(n) \qquad (34)$$

Where:

$\mathbf{C}(n)$ Is at time varying $q \times p$ matrix

$\mathbf{v}(n)$ Is zero mean white noise process and independent of $\mathbf{w}(n)$

- For $\mathbf{v}(n)$, we have also:

$$E\left\{\mathbf{v}(n)\mathbf{v}^H(k)\right\} = \begin{cases} \mathbf{Q}_v(n) & ; \ k = n \\ 0 & ; \ k \neq n \end{cases} \qquad (35)$$

- For general case of non stationary process, the optimum estimate of $\mathbf{x}(n)$ can be given by:

$$\hat{\mathbf{x}}(n) = \mathbf{A}(n-1)\hat{\mathbf{x}}(n-1) + \mathbf{K}(n)\left[\mathbf{y}(n) - \mathbf{C}(n)\mathbf{A}(n-1)\hat{\mathbf{x}}(n-1)\right] \qquad (36)$$

Where: $\mathbf{K}(n)$ Is **Kalman Gain Matrix**

Discrete Kalman Filter

- With choosing appropriate Kalman gin matrix, $\mathbf{K}(n)$, the recursion given in the previous equation, eq(36), is **Discrete Kalman Filter**
- Therefore, it is important to find the optimum $\mathbf{K}(n)$ which can minimize the mean-square estimation error
- In the next development of discrete Kalman Filter, we will assume:

$$\mathbf{A}(n), \mathbf{C}(n), \mathbf{Q}_w(n), \mathbf{Q}_v(n) \quad \Rightarrow$$

are known terms

- Denoting:

$\hat{\mathbf{x}}(n/n) \rightarrow$ The best Linear estimate of $\mathbf{x}(n)$ at time n given the observation $\mathbf{y}(i)$ for $i = 1, 2, \ldots, n$

$\hat{\mathbf{x}}(n/n-1) \rightarrow$ The best Linear estimate given the observation up to the time $n-1$

$\mathbf{e}(n/n)$ & $\mathbf{e}(n/n-1) \rightarrow$ Are the corresponding state estimation errors

Discrete Kalman Filter

- The two state estimation errors given by:

$$\mathbf{e}(n/n) = \mathbf{x}(n) - \hat{\mathbf{x}}(n/n) \qquad (37)$$

$$\mathbf{e}(n/n-1) = \mathbf{x}(n) - \hat{\mathbf{x}}(n/n-1) \qquad (38)$$

- The error covariance matrices , $\mathbf{P}(n/n)$ and $\mathbf{P}(n/n-1)$, are also given by:

$$\mathbf{P}(n/n) = E\left\{\mathbf{e}(n/n)\mathbf{e}^H(n/n)\right\} \qquad (39)$$

$$\mathbf{P}(n/n-1) = E\left\{\mathbf{e}(n/n-1)\mathbf{e}^H(n/n-1)\right\} \qquad (40)$$

- For simplicity, let’s consider we are given an estimate $\hat{\mathbf{x}}(0/0)$ of state $\mathbf{x}(0)$ and the error covariance matrix $\mathbf{P}(0/0)$
- When the observation $\mathbf{y}(1)$ is available, $\hat{\mathbf{x}}(1/1)$ can be obtained by updating $\hat{\mathbf{x}}(0/0)$

Discrete Kalman Filter

- Then the minimum mean-square error at $n = 1$ is given by:

$$\varepsilon(1) = E\left\{\|\mathbf{e}(1/1)\|^2\right\} = tr\left\{\mathbf{P}(1/1)\right\} = \sum_{i=0}^{p-1} E\left\{|\mathbf{e}_i(1/1)|^2\right\} \qquad (41)$$

- After obtaining $\hat{\mathbf{x}}(1/1)$, $\mathbf{P}(1/1)$ will be found and the estimation will continue for the next observation $\mathbf{y}(2)$ and the process continues as follows:

Given $\hat{\mathbf{x}}(n-1/n-1)$ & $\mathbf{P}(n-1/n-1)$ for $n > 0$

Step-1

Find $\hat{\mathbf{x}}(n/n-1)$

Step-2: **When new observation, $\mathbf{y}(n)$ is available**

Find minimum mean-square error estimate $\hat{\mathbf{x}}(n/n)$ of $\mathbf{x}(n)$

Discrete Kalman Filter

- In Step-1, There is no new observation and we know only $\mathbf{x}(n)$ evolves as follows:

$$\mathbf{x}(n) = \mathbf{A}(n-1)\mathbf{x}(n-1) + \mathbf{w}(n) \qquad (42)$$

- And $\hat{\mathbf{x}}(n/n-1)$ can be obtained as:

$$\hat{\mathbf{x}}(n/n-1) = \mathbf{A}(n-1)\hat{\mathbf{x}}(n-1/n-1) \qquad (43)$$

- The estimation error of eq(43) is given by:

$$\begin{aligned}\mathbf{e}(n/n-1) &= \mathbf{x}(n) - \hat{\mathbf{x}}(n/n-1)\\ &= \mathbf{A}(n-1)\mathbf{x}(n-1) + \mathbf{w}(n) - \mathbf{A}(n-1)\hat{\mathbf{x}}(n-1/n-1)\\ &= \mathbf{A}(n-1)\left[\mathbf{x}(n-1) - \hat{\mathbf{x}}(n-1/n-1)\right] + \mathbf{w}(n)\\ &= \mathbf{A}(n-1)\mathbf{e}(n-1/n-1) + \mathbf{w}(n) \qquad (44)\end{aligned}$$

Discrete Kalman Filter

- If $\hat{\mathbf{x}}(n-1/n-1)$ is the unbiased estimator of $\mathbf{x}(n-1)$, we have:

$$E\left\{\mathbf{x}(n-1)-\hat{\mathbf{x}}(n-1/n-1)\right\}=E\left\{\mathbf{e}(n-1/n-1)\right\}=0 \qquad (45)$$

- Then, $\hat{\mathbf{x}}(n/n-1)$ will be an unbiased estimate of $\mathbf{x}(n)$:

$$\begin{aligned}E\left\{\mathbf{x}(n)-\hat{\mathbf{x}}(n/n-1)\right\}&=E\left\{\mathbf{e}(n/n-1)\right\}\\&=E\left\{\mathbf{A}(n-1)\mathbf{e}(n-1/n-1)+\mathbf{w}(n)\right\}\\&=\mathbf{A}(n-1)E\left\{\mathbf{e}(n-1/n-1)\right\}+E\left\{\mathbf{w}(n)\right\} \qquad (46)\\&=0\end{aligned}$$

Zero mean white noise

- Since $\mathbf{e}(n-1/n-1)$ uncorrelated with $\mathbf{w}(n)$:

$$\mathbf{P}(n/n-1)=\mathbf{A}(n-1)\mathbf{P}(n-1/n-1)\mathbf{A}^{H}(n-1)+\mathbf{Q}_{w}(n) \qquad (47)$$

Discrete Kalman Filter

- In Step-2, the observation $\mathbf{y}(n)$ will be incorporated in to the estimate $\hat{\mathbf{x}}(n/n-1)$ to compute the linear estimation of $\mathbf{x}(n)$ based on $\hat{\mathbf{x}}(n/n-1)$ & $\mathbf{y}(n)$:

$$\hat{\mathbf{x}}(n/n) = \mathbf{K}^{'}(n)\hat{\mathbf{x}}(n/n-1) + \mathbf{K}(n)\mathbf{y}(n) \qquad (48)$$

- In eq (48), $\mathbf{K}^{'}(n)$ and $\mathbf{K}(n)$ are matrices to be specified
- The following two constraints are imposed on $\hat{\mathbf{x}}(n/n)$:

I. Unbiased: $E\left\{\mathbf{e}(n/n)\right\} = 0$

II. Minimize mean-square error $E\left\{\left\|\mathbf{e}(n/n)\right\|^{2}\right\}$

- Using eq (48), we may write $\mathbf{e}(n/n)$ as follows:

$$\begin{aligned}\mathbf{e}(n/n) &= \mathbf{x}(n) - \mathbf{K}^{'}(n)\hat{\mathbf{x}}(n/n-1) - \mathbf{K}(n)\mathbf{y}(n) \\ &= \mathbf{x}(n) - \mathbf{K}^{'}(n)\left[\mathbf{x}(n) - \mathbf{e}(n/n-1)\right] - \mathbf{K}(n)\left[\mathbf{C}(n)\mathbf{x}(n) + \mathbf{v}(n)\right] \\ &= \left[\mathbf{I} - \mathbf{K}^{'}(n) - \mathbf{K}(n)\mathbf{C}(n)\right]\mathbf{x}(n) + \mathbf{K}^{'}(n)\mathbf{e}(n/n-1) - \mathbf{K}(n)\mathbf{v}(n) \qquad (49)\end{aligned}$$

Discrete Kalman Filter

- For $\hat{\mathbf{x}}(n/n)$ to be unbiased estimator requirement, it should satisfy the following:

$$E\left\{\mathbf{e}(n/n)\right\}=0 \qquad (50)$$

- From eq (49), we can compute $E\left\{\mathbf{e}(n/n)\right\}$ as:

$$E\left\{\mathbf{e}(n/n)\right\}=E\left\{\left[\mathbf{I}-\mathbf{K}^{'}(n)-\mathbf{K}(n)\mathbf{C}(n)\right]\mathbf{x}(n)+\mathbf{K}^{'}(n)\mathbf{e}(n/n-1)-\mathbf{K}(n)\mathbf{v}(n)\right\}$$

$$=\left[\mathbf{I}-\mathbf{K}^{'}(n)-\mathbf{K}(n)\mathbf{C}(n)\right]E\left\{\mathbf{x}(n)\right\}+\mathbf{K}^{'}(n)E\left\{\mathbf{e}(n/n-1)\right\}-\mathbf{K}(n)E\left\{\mathbf{v}(n)\right\}$$

$$=\left[\mathbf{I}-\mathbf{K}^{'}(n)-\mathbf{K}(n)\mathbf{C}(n)\right]E\left\{\mathbf{x}(n)\right\} \qquad (51)$$

Equal to Zero

- From eq (51), to satisfy the requirement: $\rightarrow E\left\{\mathbf{e}(n/n)\right\}=0$

$$\mathbf{K}^{'}(n)=\mathbf{I}-\mathbf{K}(n)\mathbf{C}(n) \qquad (52)$$

Discrete Kalman Filter

- Now, we can write $\hat{\mathbf{x}}(n/n)$ incorporating the constraint given in eq(52) as:

$$\hat{\mathbf{x}}(n/n) = \left[\mathbf{I} - \mathbf{K}(n)\mathbf{C}(n)\right]\hat{\mathbf{x}}(n/n-1) + \mathbf{K}(n)\mathbf{y}(n)$$

$$= \hat{\mathbf{x}}(n/n-1) + \mathbf{K}(n)\left[\mathbf{y}(n) - \mathbf{C}(n)\hat{\mathbf{x}}(n/n-1)\right] \qquad (53)$$

- Using eq (49 & 52), the error $\mathbf{e}(n/n)$ can also be written as:

$$\mathbf{e}(n/n) = \mathbf{K}^{'}(n)\mathbf{e}(n/n-1) - \mathbf{K}(n)\mathbf{v}(n)$$

$$= \left[\mathbf{I} - \mathbf{K}(n)\mathbf{C}(n)\right]\mathbf{e}(n/n-1) - \mathbf{K}(n)\mathbf{v}(n) \qquad (54)$$

- From the fact that $\mathbf{v}(n)$ is uncorrelated $\mathbf{w}(n), \mathbf{x}(n),\ and\ \hat{\mathbf{x}}(n/n-1)$, hence; $\mathbf{e}(n/n-1) = \mathbf{x}(n) - \mathbf{x}(n/n-1)$ is also uncorrelated with $\mathbf{v}(n)$

$$E\left\{\mathbf{e}(n/n-1)v(n)\right\} = 0 \qquad (55)$$

Discrete Kalman Filter

- The error covariance matrix for $\mathbf{e}(n/n)$ can also be written as:

$$\mathbf{P}(n/n) = E\left\{\mathbf{e}(n/n)\mathbf{e}^H(n/n)\right\}$$

$$= \left[\mathbf{I} - \mathbf{K}(n)\mathbf{C}(n)\right]\mathbf{P}(n/n-1)\left[\mathbf{I} - \mathbf{K}(n)\mathbf{C}(n)\right]^H + \mathbf{K}(n)\mathbf{Q}_v(n)\mathbf{K}^H(n) \quad (56)$$

- Next we need to find the suitable value of Kalman gain $\mathbf{K}(n)$ which can minimize the mean-square error, $\xi(n)$:

$$\xi(n) = tr\left\{\mathbf{P}(n/n)\right\} \quad (57)$$

- From optimization theory, the value of $\mathbf{K}$ that makes eq(57) minimum can be found by solving the following equation

$$\frac{d\xi(n)}{d\mathbf{K}} = \frac{d}{d\mathbf{K}} tr\left\{\mathbf{P}(n/n)\right\} = 0$$

$$= \frac{d}{d\mathbf{K}} tr\left(\left[\mathbf{I} - \mathbf{K}(n)\mathbf{C}(n)\right]\mathbf{P}(n/n-1)\left[\mathbf{I} - \mathbf{K}(n)\mathbf{C}(n)\right]^H - \mathbf{K}(n)\mathbf{Q}_v(n)\mathbf{K}^H(n)\right) = 0 \quad (58)$$

Discrete Kalman Filter

- Eq(58) can be solved by applying the following characteristics of matrix differentiation :

$$\frac{d}{d\mathbf{K}} tr\{\mathbf{KA})\} = \mathbf{A}^{H} \qquad (59)$$

And

$$\frac{d}{d\mathbf{K}} tr\left\{\mathbf{KAK}^{H})\right\} = 2\mathbf{KA} \qquad (60)$$

- Then, applying the relations given in eq(59 & 60), we will have:

$$\frac{d}{d\mathbf{K}} tr\{\mathbf{P}(n/n)\} = -2[\mathbf{I}-\mathbf{K}(n)\mathbf{C}(n)]\mathbf{P}(n/n-1)\mathbf{C}^{H}(n)+2\mathbf{K}(n)\mathbf{Q}_{v}(n)=0 \qquad (61)$$

- Solving for $\mathbf{K}(n)$ using eq (61):

$$\boxed{\mathbf{K}(n)=\mathbf{P}(n/n-1)\mathbf{C}^{H}(n)\left[\mathbf{C}(n)\mathbf{P}(n/n-1)\mathbf{C}^{H}(n)+\mathbf{Q}_{v}(n)\right]^{-1}} \qquad (62)$$

Discrete Kalman Filter

- Once the Kalman gain vector is computed, the error covariance equation can be simplified by rewriting eq(56) as follows:

$$\mathbf{P}(n/n)=\left[\mathbf{I}-\mathbf{K}(n)\mathbf{C}(n)\right]\mathbf{P}(n/n-1)$$
$$-\left\{\left[\mathbf{I}-\mathbf{K}(n)\mathbf{C}(n)\right]\mathbf{P}(n/n-1)\mathbf{C}^{H}(n)+\mathbf{K}(n)\mathbf{Q}_{v}(n)\right\}\mathbf{K}^{H}(n) \qquad (63)$$

- From eq(61), it is noted that the second term of eq(63) is equal to zero, then:

$$\mathbf{P}(n/n)=\left[\mathbf{I}-\mathbf{K}(n)\mathbf{C}(n)\right]\mathbf{P}(n/n-1) \qquad (64)$$

- We have so far formulated the Kalman filter for recursive estimation of the state vector $\mathbf{x}(n)$
- To complete the recursion, what required from us is determining how the recursion should be initialized at time $n=0$

Discrete Kalman Filter

- It is noted that the initial state is unknown due to the absence of any observed data at time $n = 0$
- Therefore, the initial sate is chosen to be:

$$\hat{\mathbf{x}}(0/0) = E\left\{\mathbf{x}(0)\right\} \qquad (65)$$

- The error covariance for the initial state is also given by:

$$\mathbf{P}(0/0) = E\left\{\mathbf{x}(0)\mathbf{x}^{H}(0)\right\} \qquad (66)$$

- The above choice for the initial state makes that $\hat{\mathbf{x}}(0/0)$ is an unbiased estimate of $\mathbf{x}(0)$
- In addition, it ensures that $\hat{\mathbf{x}}(n/n)$ will be unbiased for all n since the Kalman filtering update equations are derived by putting constraint on $\hat{\mathbf{x}}(n/n)$ to be unbiased

Discrete Kalman Filter

- As we have seen from the derivation of the discrete Kalman filter, it is worth to note that Kalman gain, $\mathbf{K}(n)$, and error covariance, $\mathbf{P}(n/n)$, do not depend on the data $\mathbf{x}(n)$ [2]
- The derivation of discrete Kalman filter is summarized as follows:

State Equation $\rightarrow \mathbf{x}(n) = \mathbf{A}(n-1)\mathbf{x}(n-1) + \mathbf{w}(n)$

Observation Equation $\rightarrow \mathbf{y}(n) = \mathbf{C}(n)\mathbf{x}(n) + \mathbf{v}(n)$

Initialization $\rightarrow \hat{\mathbf{x}}(0/0) = E\left\{\mathbf{x}(0/0)\right\}$

$$\mathbf{P}(0/0) = E\left\{\mathbf{x}(0)\mathbf{x}^{H}(0)\right\}$$

Computation $\rightarrow$ *for* $n = 1, 2, \ldots$ *compute*

$$\mathbf{x}(n/n-1) = \mathbf{A}(n-1)\hat{\mathbf{x}}(n-1/n-1)$$

$$\mathbf{P}(n/n-1) = \mathbf{A}(n-1)\mathbf{P}(n-1/n-1)\mathbf{A}^{H}(n-1) + \mathbf{Q}_{w}(n)$$

$$\mathbf{K}(n) = \mathbf{P}(n/n-1)\mathbf{C}^{H}(n)\left[\mathbf{C}(n)\mathbf{P}(n/n-1)\mathbf{C}^{H}(n) + \mathbf{Q}_{v}(n)\right]^{-1}$$

$$\hat{\mathbf{x}}(n/n) = \hat{\mathbf{x}}(n/n-1) + \mathbf{K}(n)\left[\mathbf{y}(n) - \mathbf{C}(n)\hat{\mathbf{x}}(n/n-1)\right]$$

$$\mathbf{P}(n/n) = \left[\mathbf{I} - \mathbf{K}(n)\mathbf{C}(n)\right]\mathbf{P}(n/n-1)$$

Discrete Kalman Filter: Example [2]

- Suppose our goal is estimating an unknown constant x using Kalman filter from a measurement which is corrupted by a noise $v(n)$ which is :
  - Uncorrelated and;
  - Zero mean white noise wit variance σ_v^2
- Since x is constant, its value does not change with time n, hence:

$$x(n) = x(n-1) \qquad (67)$$

- The corrupted observation or the measurement, $y(n)$, is given by:

$$y(n) = x(n) + v(n) \qquad (68)$$

- From eq(67 & 68), it is noted that:

$$\mathbf{A}(n-1) = 1, \mathbf{C}(n) = 1, \mathbf{Q}_w(n) = 0, \mathbf{Q}_v(n) = \sigma_v^2$$

Discrete Kalman Filter: Example

- Since $x(n)$ is scalar, the error covariance will be also scalar and given by:

$$P(n/n) = E\left\{e^2(n/n)\right\} \qquad (69)$$

Where:

$$e(n/n) = x(n) - \hat{x}(n/n) \qquad (70)$$

- From eq(47), we can have:

$$P(n/n-1) = A(n-1)P(n-1/n-1)A^H(n-1) + Q_w(n)$$

$$= P(n-1/n-1) \qquad (71)$$

- For simplicity, we will use the notation $P(n-1)$ to represent both $P(n/n-1)$ and $P(n-1/n-1)$

Discrete Kalman Filter: Example

- From equation of Kalman gain given at eq(62), we have:

$$K(n) = P(n/n-1)\mathbf{C}^H(n)\left[\mathbf{C}(n)P(n/n-1)\mathbf{C}^H(n) + \mathbf{Q}_v(n)\right]^{-1}$$

$$= P(n/n-1)\left[P(n/n-1) + \sigma_v^2\right]^{-1}$$

$$= P(n-1)\left[P(n-1) + \sigma_v^2\right]^{-1} \qquad (72)$$

- From the update equation for $P(n/n)$ given at eq(64), we have:

$$P(n/n) = \left[\mathbf{I} - K(n)\mathbf{C}(n)\right]P(n/n-1) \qquad (73)$$

- Denoting $P(n/n)$ by represent $P(n)$, eq(73) becomes:

$$P(n) = \left[1 - K(n)\right]P(n-1) \qquad (74)$$

Discrete Kalman Filter: Example

- Substituting eq(72) for $K(n)$ in eq(74), we will have:

$$P(n) = \left[1 - K(n)\right]P(n-1)$$

$$= \left[1 - \left(P(n-1)\left[P(n-1) + \sigma_v^2\right]^{-1}\right)\right]P(n-1)$$

$$= \left[1 - \frac{P(n-1)}{P(n-1) + \sigma_v^2}\right]P(n-1)$$

$$= \frac{P(n-1)\sigma_v^2}{P(n-1) + \sigma_v^2} \qquad (75)$$

Discrete Kalman Filter: Example

- We can solve eq(75) recursively as:

$$P(1) = \frac{P(0)\sigma_v^2}{P(0)+\sigma_v^2}$$

$$P(2) = \frac{P(1)\sigma_v^2}{P(1)+\sigma_v^2} = \frac{P(0)\sigma_v^2}{2P(0)+\sigma_v^2}$$

$$P(3) = \frac{P(2)\sigma_v^2}{P(2)+\sigma_v^2} = \frac{P(0)\sigma_v^2}{3P(0)+\sigma_v^2}$$

$$\vdots$$

- Hence we can write $P(n)$ in general form as :

$$P(n) = \frac{P(0)\sigma_v^2}{nP(0)+\sigma_v^2} \qquad (76)$$

Discrete Kalman Filter: Example

- From eq(72) and eq(75), we have :

$$K(n) = \frac{P(n)}{\sigma_v^2} = \frac{1}{\sigma_v^2}\left[\frac{P(0)\sigma_v^2}{nP(0)+\sigma_v^2}\right]$$

$$= \frac{P(0)}{nP(0)+\sigma_v^2} \qquad (77)$$

- Then using eq(53), we will have:

$$\hat{x}(n) = \hat{x}(n-1) + K(n)\left[y(n) - \mathbf{C}(n)\hat{x}(n-1)\right]$$

$$= \hat{x}(n-1) + \frac{P(0)}{nP(0)+\sigma_v^2}\left[y(n) - \hat{x}(n-1)\right] \qquad (78)$$

Summary

- **Discrete Kalman Filter:**
  - ✓ Optimum filter for estimating and predicting non stationary process
- **State Transition Matrix**
  - ✓ Maps the current state to the next state
  - ✓ Propagates state estimation forward in time
- **Error Covariance Matrix**
  - ✓ Track Variance (mean squared error) of each state and
  - ✓ the error correlation between different states
- **The Kalman gain**
  - ✓ An optimal adaptive gain which minimizes the estimation error

References

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Thank You!