Statistical Digital Signal Processing

Week 9 Non- Parametric Spectrum Estimation: Periodogram Method

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Previous Topic (Week-8)

Discrete Kalman Filter

- Basics Discrete Kalman Filter
- Discrete Kalman Filter development and Analysis
- Kalman Gain, State Transition Matrix, and Error covariance matrix

Lecture Learning Outcomes

1. Explain the fundamental concepts and objectives of spectrum estimation in signal processing
2. Describe the periodogram method and outline the steps involved in estimating the power spectral density of a signal
3. Analyze the overall performance of the periodogram, including its strengths and practical limitations
4. Examine the concept of periodogram bias and its impact on the accuracy of spectral estimates
5. Evaluate the variance of the periodogram and discuss why it leads to inconsistency in spectral estimation

Week 9: Periodogram Method

Outline

- Spectrum Estimation: Introduction
- Periodogram Method
- Performance of Periodogram
- Periodogram Bias
- Periodogram Variance

Spectrum Estimation: Introduction

- On topics we have seen so far, such as signal modeling and optimum filtering, most of the linear equations to be solved are based on autocorrelation functions.
- However, the autocorrelation of the signal may not be known and we may be required to estimate the autocreation and the power spectrum the stationary random signal
- In this session, we will focus on estimating the power spectral density of a wide sense stationary random process
- Since the power spectrum is Fourier transform of the autocorrelation sequence, estimating the power spectrum is equivalent to estimating the autocorrelation
- For an autocorrelation ergodic process is given by [1]:

$$r_x(k) = \lim_{N\to\infty}\left\{\frac{1}{2N+1}\sum_{n=-N}^{N} x(n+k)x^*(n)\right\} = E\{x(n+k)x^*(n)\} \qquad (1)$$

Spectrum Estimation: Introduction

- As revealed by eq(1), if $x(n)$ is known for all n, finding the power spectrum is easy and straight forward
- However, there is a challenge since the number of data we may have is limited and few in most cases
- This limitation may stem from the nature of the data collection process
    - **Example:** Seismic data analysis from earthquake, where signals last only a short time
- However, in some scenarios, the requirement of limited data set is imposed by the spectral characteristics of the process
    - **Example:** in speech signal, the stationarity requirement limit the length the data set in which the signal assumed nearly stationary
- The other challenge is, the data is often extremely corrupted by noise or contaminated by interference

Spectrum Estimation: Introduction

- Therefore, the power spectrum estimation problem becomes estimating $P_x(e^{j\omega})$ from finite noisy data measured from $x(n)$
- Spectrum estimation problem is important in a variety of engineering applications
- For example, recalling the transfer function non-causal smoothing filter, it was given by:

$$H(e^{j\omega}) = \frac{P_d(e^{j\omega})}{P_d(e^{j\omega}) + P_v(e^{j\omega})} \qquad (2)$$

Where:

$P_d(e^{j\omega})$ is the power spectrum of the desired output, $d(n)$, of the Wiener filter

$P_v(e^{j\omega})$ is the power spectrum of the noise signal, $v(n)$,

- Therefore, to design the non causal IIR smoothing Wiener filter, determining the power spectrum of both $d(n)$ and $v(n)$ is required

Spectrum Estimation: Introduction

- In general, there are **two broad categories** of spectrum estimation techniques;

I. Classical or Nonparametric Methods

- Based on estimating the autocorrelation sequence from a given set of data and;
- Estimating the power spectrum by taking the Fourier transform of the estimated autocorrelation sequence

II. Nonclassical or Parametric Methods

- Based on using a model for the process to estimate the power spectrum
- For example, suppose $x(n)$ is p^th^ order AR process, measured value of $x(n)$ can be used to estimate the model parameter $a_p(k)$
- Then the power spectrum can be estimated from the model as:

$$\hat{P}_x(e^{j\omega}) = 1 \Bigg/ \left| \sum_{k=0}^{p} \hat{a}_p(k) z^{-k} \right|^2 \qquad (3)$$

Spectrum Estimation: Introduction

- In this and the next two consecutive weeks, we will focus on the Classical or Nonparametric Methods including:

  - **Periodogram Method**
  - **Modified Periodogram Method**
  - **Bartlett's Method**
  - **Welch's Method**
  - **Blackman-Tukey Method**

- In today's session, we will focus on one of the Classical or Nonparametric Methods, Which is called **Periodogram Method**

Periodogram Method

- The power spectrum of wide sense stationary process is given by the Fourier transform of the autocorrelation sequence as:

$$P_x(e^{j\omega}) = \sum_{k=-\infty}^{\infty} r_x(k) e^{-jk\omega} \qquad (4)$$

- As recalled from eq(1), for an autocorrelation ergodic process, we have :

$$r_x(k) = \lim_{N\to\infty} \left\{ \frac{1}{2N+1} \sum_{n=-N}^{N} x(n+k)x^*(n) \right\} \qquad (5)$$

- However, suppose $x(n)$ measured in finite interval, $n = 0,1,\ldots,N-1$, the autocorrelation can be estimated as:

$$\hat{r}_x(k) = \frac{1}{N} \sum_{n=0}^{N-1} x(n+k)x^*(n) \qquad (6)$$

Periodogram Method

- Excluding the values of $x(n)$ falling outside $n = 0, 1, \ldots, N-1$, eq(6) simplified to:

$$\hat{r}_x(k) = \frac{1}{N} \sum_{n=0}^{N-1-k} x(n+k)x^*(n) \qquad (7)$$

- It is noted that $\hat{r}_x(-k) = \hat{r}_x^*(k)$ and $\hat{r}_x(k) = 0$ for $|k| \geq N$
- Taking the DTFT of $\hat{r}_x(k)$, we will obtain the estimated power spectrum or the ***Periodogram***, $\hat{P}_{per}(e^{j\omega})$, as:

$$\hat{P}_{per}(e^{j\omega}) = \sum_{k=-\infty}^{\infty} \hat{r}_x(k)e^{-jk\omega} = \sum_{k=-N+1}^{N-1} \hat{r}_x(k)e^{-jk\omega} \qquad (8)$$

- However ,The periodogram in eq(8) is written based on the autocorrelation, it is more convenient to write it in terms of the process $x(n)$

Periodogram Method

- Let define a finite length signal, $x_N(n)$, as follows:

$$x_N(n) = \begin{cases} x(n) & ; \ 0 \le n < N \\ 0 & ; \ Otherwise \end{cases} \qquad (9)$$

- Alternatively $x_N(n)$ can be written as:

$$x_N(n) = w_R(n)x(n) \qquad (10)$$

Where: $w_R(n) = \begin{cases} 1 & ; \ 0 \le n < N \\ 0 & ; \ Otherwise \end{cases}$ is a rectangular window

- Writing the estimated autocorrelation in terms of $x_N(n)$:

$$\hat{r}_x(k) = \frac{1}{N}\sum_{n=0}^{N-1} x_N(n+k)x_N^*(n) = \frac{1}{N} x_N(k) * x_N^*(-k) \qquad (11)$$

Periodogram Method

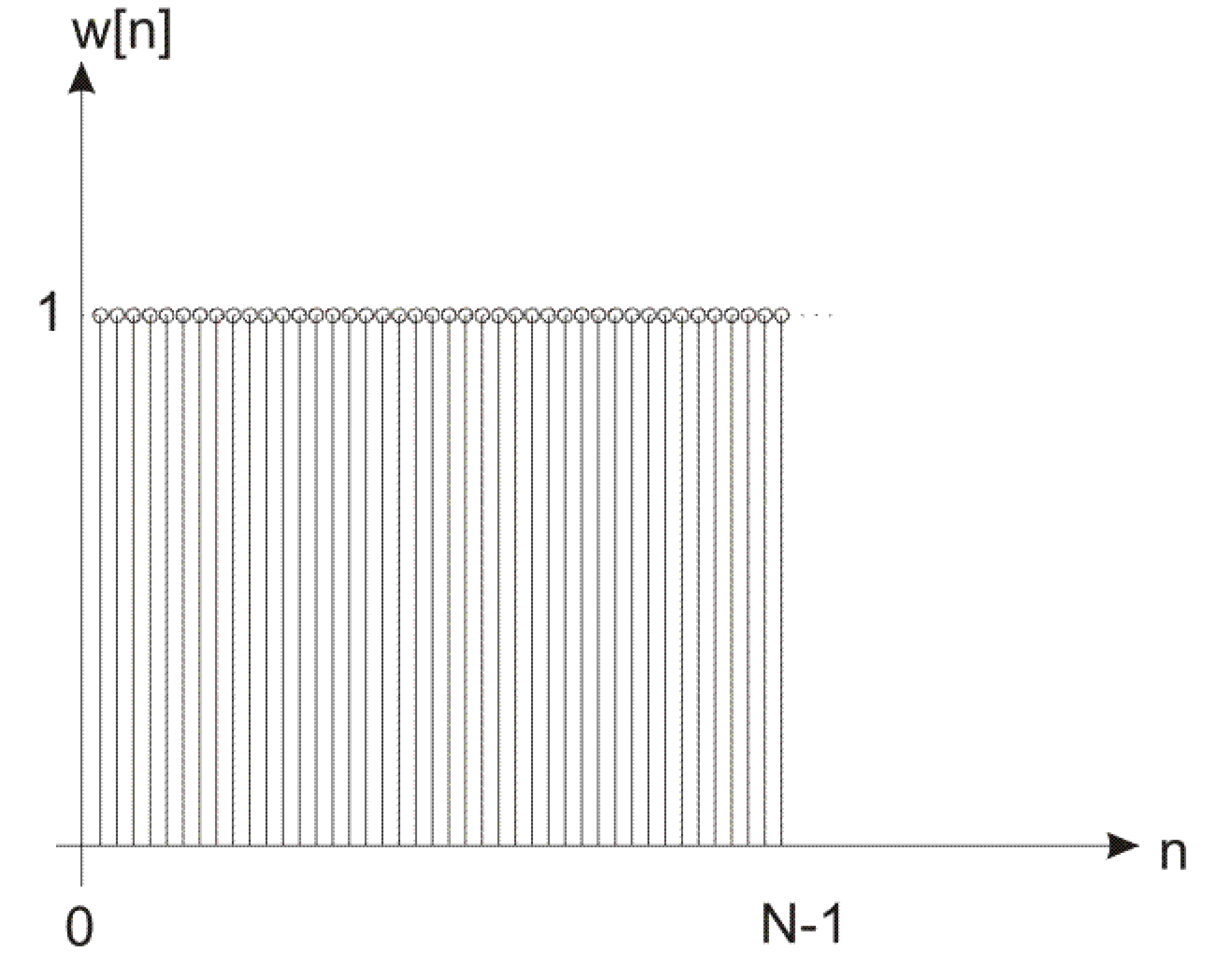

Figure 1: Rectangular Window

Source: “Window Functions,” MIKROE. https://cdn.mikroe.com/ebooks/img/8/2016/02/digital-filter-design-chapter-02-image-3-2.gif

Periodogram Method

- Taking the Fourier transform of eq(11), the periodogram becomes :

$$\hat{P}_{per}(e^{j\omega}) = DTFT\ \ \hat{r}_x(k)\ = DTFT\left\{\frac{1}{N}x_N(k) * x_N^*(-k)\right\}$$

$$= \frac{1}{N}X_N(e^{j\omega})X_N^*(e^{j\omega})$$

$$= \frac{1}{N}\left|X_N(e^{j\omega})\right|^2 \qquad (12)$$

Where:

$$X_N(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x_N(n)e^{-jn\omega} = \sum_{n=0}^{N-1} x(n)e^{-jn\omega} \qquad (13)$$

Performance of Periodogram

- In this section we will focus on the mean square convergence of the periodogram.
- We will be interested in whether or not the following equality condition satisfied:

$$\lim_{N\to\infty} E\left\{\left[\hat{P}_{per}(e^{j\omega}) - P_x(e^{j\omega})\right]^2\right\} = 0 \qquad (14)$$

- For a periodogram to be convergent in mean square sense, it is required to be asymptotically unbiased and consistent estimator:
- Hence it should satisfy the following:

  - **Asymptotical Unbiased** ➡ $$\lim_{N\to\infty} E\left\{\hat{P}_{per}(e^{j\omega})\right\} = P_x(e^{j\omega}) \qquad (15)$$
  - **Consistent Estimate** ➡ $$\lim_{N\to\infty} Var\left\{\hat{P}_{per}(e^{j\omega})\right\} = 0 \qquad (16)$$

Performance of Periodogram : Periodogram Bias

- We begin by calculating the expected value of the estimated autocorrelation sequence to compute the bias of the periodogram as:

$$E\{\hat{r}_x(k)\} = \frac{1}{N}\sum_{n=0}^{N-1-k} E\left[x(n+k)x^*(n)\right] = \frac{1}{N}\sum_{n=0}^{N-1-k} r_x(k) = \frac{N-k}{N} r_x(k) \quad (17)$$

- **Note:** for $k \geq N$ the expected value in eq(17) is equal to zero
- Using the conjugate symmetry of $\hat{r}_x(k)$ eq (17) can also be written as:

$$E\{\hat{r}_x(k)\} = w_b(k) r_x(k) \quad (18)$$

Where:

Bartlett or Triangular Window

$$w_b(k) = \begin{cases} \frac{N-|k|}{N} & ; \ |k| \leq N \\ 0 & ; \ |k| > N \end{cases} \quad (19)$$

Performance of Periodogram : Periodogram Bias

- Then the expected value of the periodogram can be calculated as:

$$E\{\hat{P}_{per}(e^{j\omega})\} = E\left\{\frac{1}{N}\sum_{k=-N+1}^{N-1}\hat{r}_x(k)e^{-jk\omega}\right\} = \frac{1}{N}\sum_{k=-N+1}^{N-1} E\ \hat{r}_x(k)\ e^{-jk\omega}$$

$$= \frac{1}{N}\sum_{k=-N+1}^{N-1} w_b(k)r_x(k)\ e^{-jk\omega} \qquad (20)$$

- Eq(20) is the Fourier transform of the product $w_b(k)r_x(k)$, therefore, the expectation of the periodogram can be written as:

$$E\{\hat{P}_{per}(e^{j\omega})\} = DTFT\ \ w_b(k)r_x(k)$$

$$= \frac{1}{2\pi}P_x(e^{j\omega}) * W_b(e^{j\omega}) \qquad (21)$$

Performance of Periodogram : Periodogram Bias

- $W_b(e^{j\omega})$ is the Fourier transform of the Bartlett window $w_b(k)$ which is given by:

$$W_b(e^{j\omega}) = \frac{1}{N}\left[\frac{\sin\ N\omega/2}{\sin\ \omega/2}\right]^2 \qquad (22)$$

- Then we need to check whether or not the asymptotically unbiased condition is satsfied :

$$\lim_{N\to\infty} E\ \hat{P}_{per}(e^{j\omega}) = \lim_{N\to\infty}\left[P_x(e^{j\omega}) * \underbrace{\frac{1}{2\pi N}\left[\frac{\sin\ N\omega/2}{\sin\ \omega/2}\right]^2}\right]$$

$$= P_x(e^{j\omega}) \qquad (23)$$

$$= \delta(e^{j\omega})\ \textbf{for}\ N \to \infty$$

- Therefore, eq(23) confirms that the **periodogram** is **asymptotically unbiased**

Performance of Periodogram : Periodogram Bias

Effect of The Bartlett Window

- To visualize the effect of, $w_b(k)$, on the expected value of the periodogram, Lets consider a random process containing random phase sinusoid and white noise
- **Example:**

$$x(n) = A\sin(n\omega + \phi) + v(n) \qquad (24)$$

Where: ϕ is a uniformly distributed random variable over $[-\pi, \pi]$

$v(n)$ is a white noise with variance σ_v^2

- The power spectrum of $x(n)$ is given by:

$$P_x(e^{j\omega}) = \sigma_v^2 + \frac{1}{2}\pi A^2 \; \delta(\omega - \omega_0) + \delta(\omega + \omega_0) \qquad (25)$$

Performance of Periodogram : Periodogram Bias

- The expected value of the periodogram is

$$E\{\hat{P}_{per}(e^{j\omega})\} = \frac{1}{2\pi} P_x(e^{j\omega}) * W_b(e^{j\omega}) = \sigma_v^2 + \frac{1}{4} A^2 \left[W_b(e^{j(\omega-\omega_0)}) + W_b(e^{j(\omega+\omega_0)}) \right] \quad (26)$$

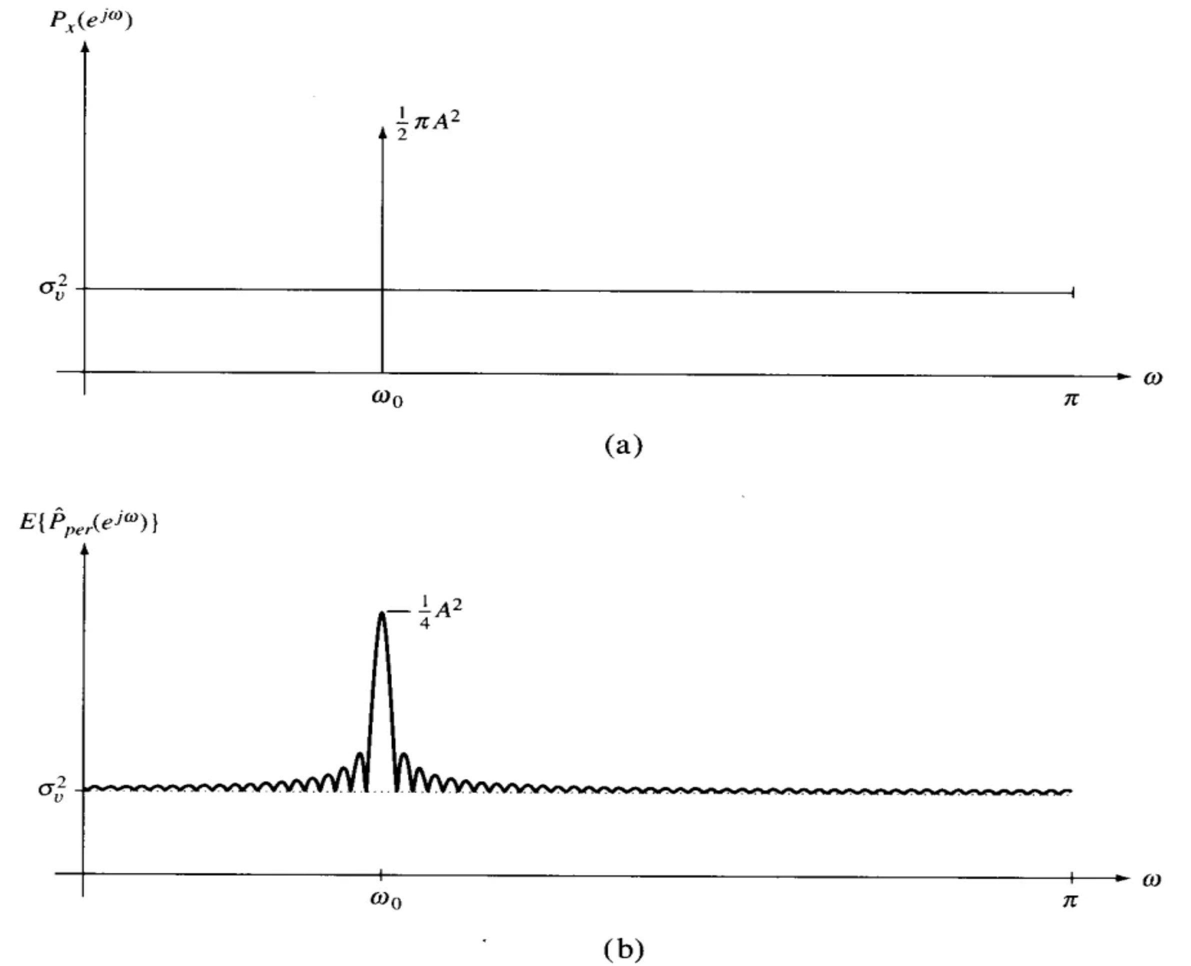

Figure 2: (a) the power spectrum of a sinusoid in presence of white noise (b) the expected value of the periodogram for N=64

Source: "Power Spectrum Estimation," Dedicated Lark. https://custom-images.strikinglycdn.com/res/hrscywv4p/image/upload/c_limit,fl_lossy,h_9000,w_1200,f_auto,q_auto/24944559/402674_71665.png

Performance of Periodogram : Periodogram Bias

- From the previous example, the following two effects of the Bartlett window have been visualized:
  - **Spectral smoothing (smearing):** the response to a signal consisting of a single frequency ω_0 is not an impulse, rather it is $W_b(e^{j(\omega-\omega_0)})$ with a 3dB bandwidth of $\approx 2\pi/N$
  - **Side Lobs:** power leakage through the side lobs creates confusion that as if the signal posses spectral component at $\omega_0 \pm \frac{2\pi}{N}k$ frequencies
- The side lob bandwidth limits the resolution ability to resolve closely spaced sinusoids
- The resolution is can be written in terms of the main lobe 3dB bandwidth as:

$$\mathbf{Res}\{\hat{P}_{per}(e^{j\omega}) = \Delta\omega = 0.89\frac{2\pi}{N} \qquad (27)$$

Performance of Periodogram : Periodogram Bias

- Periodogram of $x(n) = 15\sin(0.24\pi n + \phi) + 15\sin(0.25\pi n + \phi)$

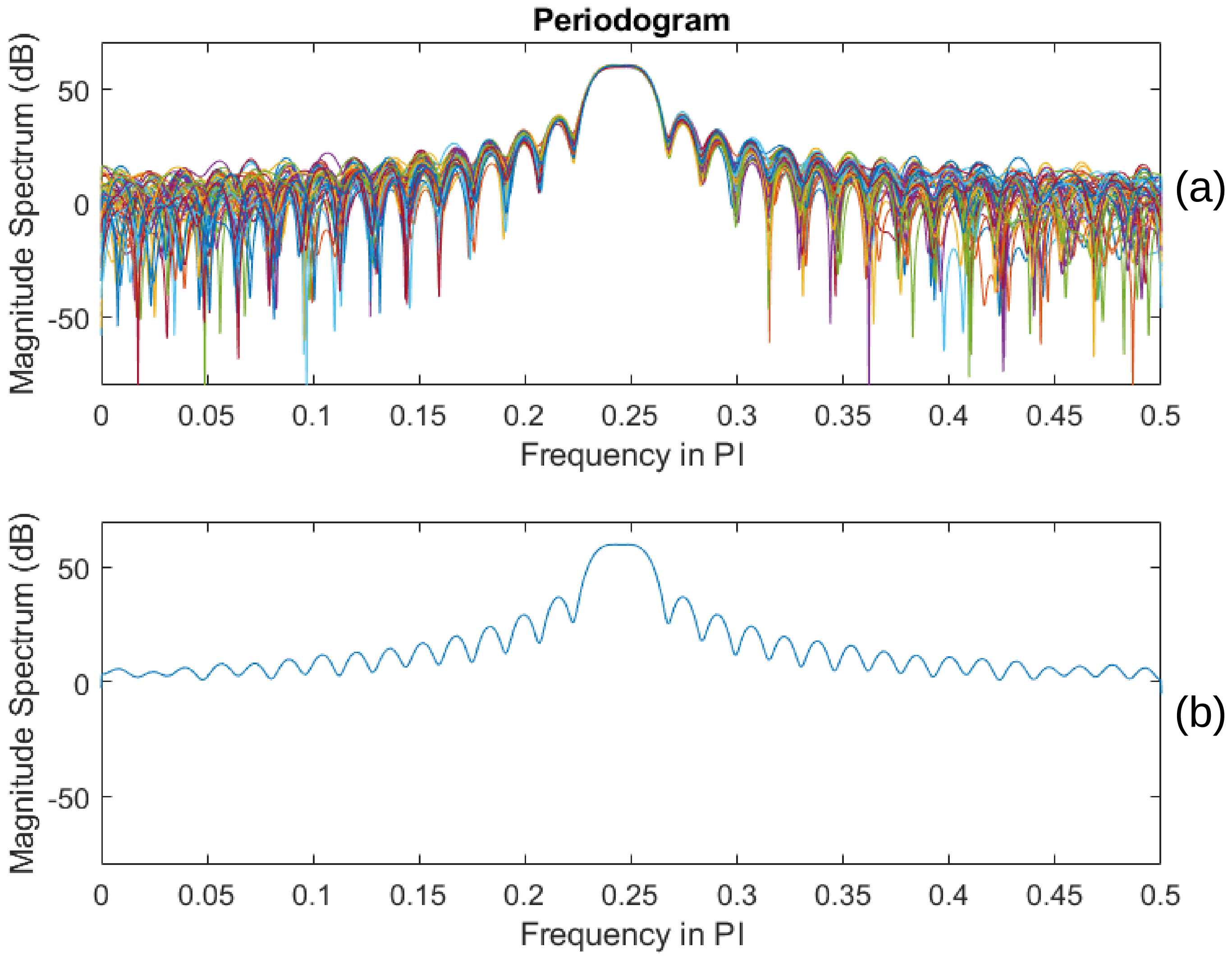

Figure 3: Periodogram of two sinusoids (a) Overlay plot of 50 periodograms using N=64 data values (b) the periodogram average

Source: MATLAB-generated plot using modified source code adapted from: S. Lakshmi, *"Spectrum-estimation,"* GitHub repository. https://github.com/srilakshmialla/Spectrum-estimation

Performance of Periodogram : Periodogram Bias

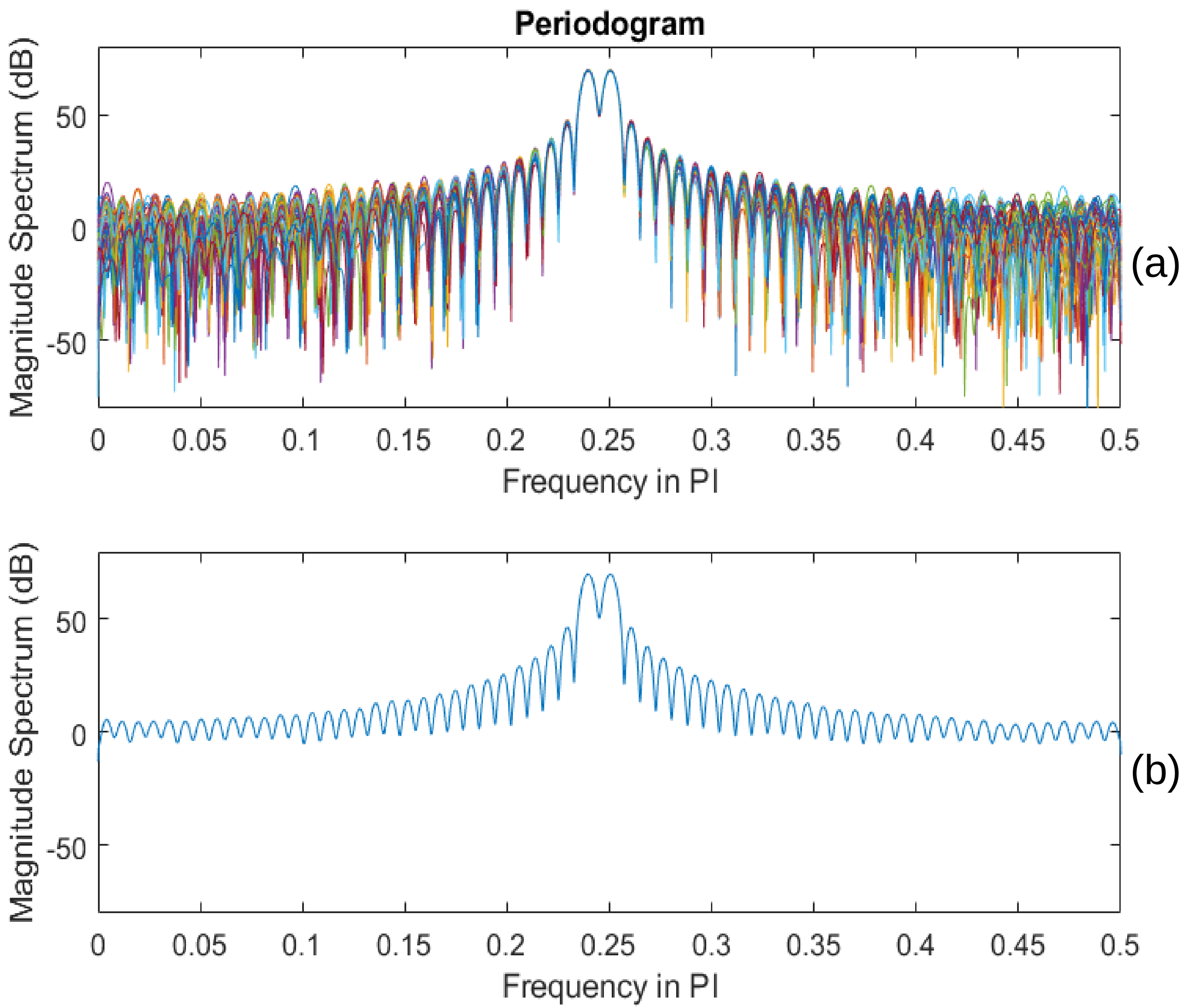

Figure 4: Periodogram of two sinusoids (a) Overlay plot of 50 periodograms using N=128 data values (b) the periodogram average

Source: MATLAB-generated plot using modified source code adapted from: S. Lakshmi, *"Spectrum-estimation,"* GitHub repository. https://github.com/srilakshmialla/Spectrum-estimation

Performance of Periodogram : Periodogram Bias

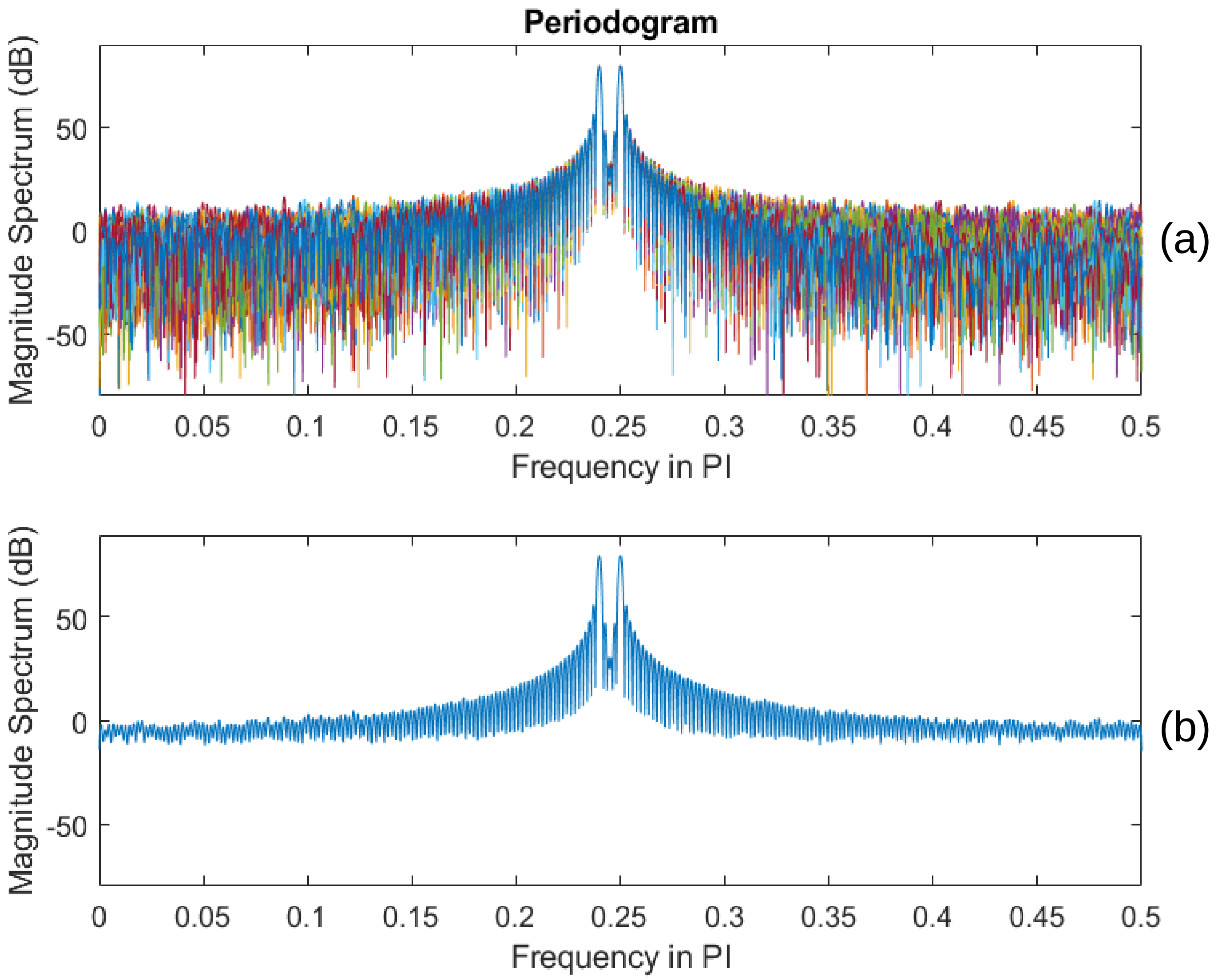

Figure 5: Periodogram of two sinusoids (a) Overlay plot of 50 periodograms using N=512 data values (b) the periodogram average

Source: MATLAB-generated plot using modified source code adapted from: S. Lakshmi, *"Spectrum-estimation,"* GitHub repository. https://github.com/srilakshmialla/Spectrum-estimation

Performance of Periodogram : Periodogram Variance

- We have proved that the periodogram is an asymptotically unbiased estimator of the power spectrum
- Now we will check whether the periodogram is a consistent estimate or not by checking the following condition:

$$\lim_{N\to\infty} Var\left\{ \hat{P}_{per}(e^{j\omega}) \right\} = 0 \qquad (28)$$

- Although proving this in the general case is difficult, we instead focus on white Gaussian noise; still challenging, but tractable
- Recalling the periodogram is given by:

$$\hat{P}_{per}(e^{j\omega}) = \frac{1}{N} X_N(e^{j\omega}) X_N^*(e^{j\omega})$$

$$= \frac{1}{N}\left[\sum_{k=0}^{N-1} x(k) e^{-jk\omega}\right]\left[\sum_{l=0}^{N-1} x^*(l) e^{jl\omega}\right]$$

Performance of Periodogram : Periodogram Variance

$$= \frac{1}{N}\sum_{k=0}^{N-1}\sum_{l=0}^{N-1} x(k)x^*(l)e^{-j(k-l)\omega} \tag{29}$$

- To compute the covariance, first evaluating the correlation between samples of the periodogram at two different points in frequency [2]:

$$E\left\{\hat{P}_{per}(e^{j\omega_1})\hat{P}_{per}(e^{j\omega_2})\right\} = E\left\{\left[\frac{1}{N}\sum_{k=0}^{N-1}\sum_{l=0}^{N-1} x(k)x^*(l)e^{-j(k-l)\omega_1}\right]\left[\frac{1}{N}\sum_{m=0}^{N-1}\sum_{n=0}^{N-1} x(m)x^*(n)e^{-j(m-n)\omega_2}\right]\right\}$$

$$= \frac{1}{N^2}\sum_{k=0}^{N-1}\sum_{l=0}^{N-1}\sum_{m=0}^{N-1}\sum_{n=0}^{N-1} E\{x(k)x^*(l)x(m)x^*(n)\}e^{-j(k-l)\omega_1}e^{-j(m-n)\omega_2} \tag{30}$$

Fourth-Order Moment of the Signal

Performance of Periodogram : Periodogram Variance

- Since $x(n)$ is assumed to be white Gaussian noise, moment factorizing theorem can be applied to simplify the fourth-order moment term in eq(30):

$$E\{x(k)x^*(l)x(m)x^*(n)\} = E\{x(k)x^*(l)\}E\{x(m)x^*(n)\} + E\{x(k)x^*(n)\}E\{x(m)x^*(l)\}$$

$$= r_x(k-l)r_x(m-n) + r_x(k-n)r_x(m-l) \qquad (31)$$

$$= \begin{cases} \sigma_x^4 & ; k = l \text{ and } m = n \\ 0 & ; otherwise \end{cases}$$

$$= \begin{cases} \sigma_x^4 & ; k = n \text{ and } l = m \\ 0 & ; otherwise \end{cases}$$

- Using the above facts, the second-order moment of the periodogram can be written as the summation of two terms

Performance of Periodogram : Periodogram Variance

- **The first term:** for $k = l$ **and** $m = n$, we have:

$$\frac{1}{N^2}\sum_{k=0}^{N-1}\sum_{m=0}^{N-1}\sigma_x^4 = \sigma_x^4 \qquad (32)$$

- **The second term:** for $k = n$ **and** $l = m$, we have:

$$\frac{1}{N^2}\sum_{k=0}^{N-1}\sum_{l=0}^{N-1}\sigma_x^4 e^{-j(k-l)\omega_1} e^{j(k-l)\omega_2} = \frac{\sigma_x^4}{N^2}\sum_{k=0}^{N-1} e^{-jk(\omega_1-\omega_2)}\sum_{l=0}^{N-1} e^{jl(\omega_1-\omega_2)}$$

$$= \frac{\sigma_x^4}{N^2}\left[\frac{1-e^{-jN(\omega_1-\omega_2)}}{1-e^{-j(\omega_1-\omega_2)}}\right]\left[\frac{1-e^{jN(\omega_1-\omega_2)}}{1-e^{j(\omega_1-\omega_2)}}\right]$$

$$= \sigma_x^4\left[\frac{\sin N(\omega_1-\omega_2)/2}{N\sin(\omega_1-\omega_2)/2}\right]^2 \qquad (33)$$

Performance of Periodogram : Periodogram Variance

- Then doing the summation of the first and the second terms of the periodogram given at eq(32) and eq(33) respectively, we will have:

$$E\left\{\hat{P}_{per}(e^{j\omega_1})\hat{P}_{per}(e^{j\omega_2})\right\} = \sigma_x^4\left[1+\left[\frac{\sin N(\omega_1-\omega_2)/2}{N\sin(\omega_1-\omega_2)/2}\right]^2\right] \quad (34)$$

- From the covariance of the periodogram, we will have also:

$$\mathbf{Cov}\left\{\hat{P}_{per}(e^{j\omega_1})\hat{P}_{per}(e^{j\omega_2})\right\} = E\left\{\hat{P}_{per}(e^{j\omega_1})\hat{P}_{per}(e^{j\omega_2})\right\} - E\left\{\hat{P}_{per}(e^{j\omega_1})\right\}E\left\{\hat{P}_{per}(e^{j\omega_2})\right\} \quad (35)$$

- From the fact that for $N \to \infty$, the expected value of the periodogram converges to the true power spectral density, σ_x^2 , the covariance can be simplified as:

Performance of Periodogram : Periodogram Variance

$$\mathbf{Cov}\left\{\hat{P}_{per}(e^{j\omega_1})\hat{P}_{per}(e^{j\omega_2})\right\} = \sigma_x^4\left[\frac{\sin N(\omega_1-\omega_2)/2}{N\sin(\omega_1-\omega_2)/2}\right]^2 \qquad (36)$$

- Then for $\omega = \omega_1 = \omega_2$, the covariance will be equal to the variance:

$$\mathbf{Var}\left\{\hat{P}_{per}(e^{j\omega})\right\} = \mathbf{Cov}\left\{\hat{P}_{per}(e^{j\omega})\hat{P}_{per}(e^{j\omega})\right\}$$

$$= \sigma_x^4 = P_x^2(e^{j\omega}) \qquad (37)$$

- Eq(37) revealed that the variance of the program does not go to zero as $N \to \infty$
- Hence Periodogram is **not consistent estimate** of the true power spectrum

Performance of Periodogram : Periodogram Variance

- However, calculating the variance of the periodogram is difficult for general case, we may drive an approximated variance
- From the scholastic signal modeling, we recall that the random signal $x(n)$ having power spectrum $P_x(e^{j\omega})$ can be modeled as an out put of LSI filter $h(n)$ with an input of unit variance white noise $v(n)$

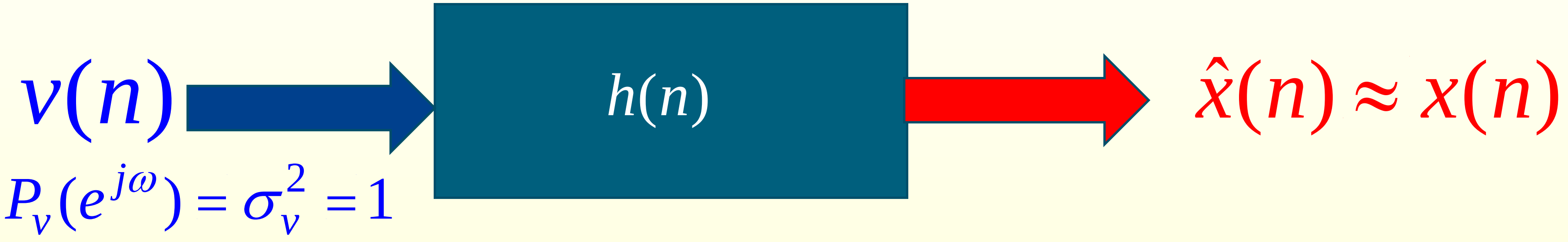

Figure 6: Linear modeling of random signal using LSI filter

- The power spectrum of the output signal is given by:

$$P_x(e^{j\omega}) = P_v(e^{j\omega})H(e^{j\omega})H^*(e^{j\omega}) = |H(e^{j\omega})|^2 \qquad (38)$$

Performance of Periodogram : Periodogram Variance

- Suppose, $x_N(n)$ and $v_N(n)$ are a sequence of length N obtained by windowing $x(n)$ and $v(n)$ respectively, the periodogram of the input-output process becomes:

$$\hat{P}_{per}^{(x)} = \frac{1}{N}\left|X_N(e^{j\omega})\right|^2 \quad (39)$$

$$\hat{P}_{per}^{(v)} = \frac{1}{N}\left|V_N(e^{j\omega})\right|^2 \quad (40)$$

- But from the linear input-output relation of LSI filter, we have:

$$x_N(n) \approx h(n) * v_N(n) \quad (41)$$

- In frequency domain, we have:

$$X_N(e^{j\omega}) \approx H(e^{j\omega})V_N(e^{j\omega}) \quad (42)$$

Performance of Periodogram : Periodogram Variance

- From eq(42), we have:

$$\left|X_N(e^{j\omega})\right|^2 \approx \left|H(e^{j\omega})\right|^2 \left|V_N(e^{j\omega})\right|^2 \tag{43}$$

- Using eq(38), eq(43) can be simplified as :

$$\left|X_N(e^{j\omega})\right|^2 \approx \left|H(e^{j\omega})\right|^2 \left|V_N(e^{j\omega})\right|^2 \approx P_x(e^{j\omega})\left|V_N(e^{j\omega})\right|^2 \tag{44}$$

- From eq(39, 40, and 44), we will obtain:

$$\hat{P}_{per}^{(x)} \approx \frac{1}{N}\left|x_N(e^{j\omega})\right|^2 \approx P_x(e^{j\omega})\left[\frac{1}{N}\left|V_N(e^{j\omega})\right|^2\right]$$

$$\approx P_x(e^{j\omega})\hat{P}_{per}^{(v)} \tag{45}$$

Performance of Periodogram : Periodogram Variance

- Then the variance can be approximated as:

$$\mathbf{Var}\ \hat{P}_{per}^{(x)} \approx \mathbf{Var}\ P_x(e^{j\omega})\hat{P}_{per}^{(v)} \approx P_x^2(e^{j\omega})\mathbf{Var}\ \hat{P}_{per}^{(v)} \approx P_x^2(e^{j\omega})\sigma_v^4$$

$$\approx P_x^2(e^{j\omega}) \qquad (46)$$

- Hence the periodogram is not consistent estimate of the power spectrum

Summary

- **Spectrum Estimation Methods:**
  - ✓ **Classical or Non-Parametric Methods:** based on the data of the signal
  - ✓ **Non-Classical or Parametric Methods:** model based

- **Non Parametric Methods**
  - ✓ **Periodogram Method**
    - Using finite length data
    - Using rectangular window function
    - Asymptotically unbiased estimator
    - Non-consistent estimator

References

[1] Monson H. Hayes, “Statistical Digital Signal Processing and Modeling’’, John Wiley and sons, Pp.391, 1996.

[2] Charles W. Therrien,‘’ Discrete Random Signals and Statistical Signal Processing’’, Prentice Hall, Pp.590, 1992.

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Thank You!