Statistical Digital Signal Processing

Week 10 Non- Parametric Spectrum Estimation: Modified Periodogram and Bartlett Methods

Lecturer: ***Zelalem Hailu (Assistant Prof.)***

Addis Ababa Science and Technology University,
Addis Ababa, Ethiopia

Previous Topic (Week-9)

Non- Parametric Spectrum Estimation: Periodogram Method

- Spectrum Estimation: Introduction
- Periodogram Method
- Performance of Periodogram
- Periodogram Bias
- Periodogram Variance

Lecture Learning Outcomes

1. Explain the concept and motivation behind the Modified Periodogram Method in spectral estimation
2. Describe the formulation and implementation of the Modified Periodogram, including the role of windowing functions
3. Analyze the performance of the Modified Periodogram, particularly in terms of bias, variance, and spectral leakage
4. Explain the principles of Bartlett’s Method and how it improves spectral estimation through periodogram averaging
5. Evaluate the performance of Bartlett’s Method, including its advantages and limitations compared to the standard periodogram

Week 10: Modified Periodogram and Bartlett Methods

Outline

- Modified Periodogram Method: Introduction
- Modified Periodogram
- Performance of Modified Periodogram
- Bartlett’s Method
- Performance of Bartlett’s Method

Modified Periodogram Method: Introduction

- In the previous session, we have seen periodogram is asymptotically unbiased but not consistent estimate of the power spectrum
- It is also recalled that the periodogram can be calculated as:

$$\hat{P}_{per}(e^{j\omega}) = \frac{1}{N}\left|X_N(e^{j\omega})\right|^2 = \frac{1}{N}\left|\sum_{n=-\infty}^{\infty} x(n)w_R(n)e^{-jn\omega}\right|^2$$

$$= \frac{1}{N}\left[\sum_{n=-\infty}^{\infty} x(n)w_R(n)e^{-jn\omega}\right]\left[\sum_{m=-\infty}^{\infty} x(m)w_R(m)e^{-jm\omega}\right]^* \qquad (1)$$

- What would happen on the performance of the estimator if we change the rectangular window, $w_R(n)$, in eq(1) by another type of window?
- To understand the effect of the window choice on the bias and variance of periodogram, let's analyze the expected value of eq(1)

Modified Periodogram Method : Introduction

$$E\left\{\hat{P}_{per}(e^{j\omega})\right\} = \frac{1}{N} E\left\{\left[\sum_{n=-\infty}^{\infty} x(n)w_R(n)e^{-jn\omega}\right]\left[\sum_{m=-\infty}^{\infty} x(m)w_R(m)e^{-jm\omega}\right]^*\right\}$$

$$= \frac{1}{N} E\left\{\sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty} x(n)x^*(m)w_R(m)w_R(n)e^{-j(n-m)\omega}\right\}$$

$$= \frac{1}{N}\sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty} E\{x(n)x^*(m)\}w_R(m)w_R(n)e^{-j(n-m)\omega}$$

$$= \frac{1}{N}\sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty} r_x(n-m)w_R(m)w_R(n)e^{-j(n-m)\omega} \quad (2)$$

Modified Periodogram Method : Introduction

- Applying change of variables, $k = n - m$, eq(2) can be simplified as:

$$E\left\{\hat{P}_{per}(e^{j\omega})\right\} = \frac{1}{N}\sum_{k=-\infty}^{\infty}\sum_{n=-\infty}^{\infty} r_x(k)w_R(n-k)w_R(n)e^{-jk\omega}$$

$$= \frac{1}{N}\sum_{k=-\infty}^{\infty} r_x(k)\left[\sum_{n=-\infty}^{\infty} w_R(n-k)w_R(n)\right]e^{-jk\omega}$$

$$= \frac{1}{N}\sum_{k=-\infty}^{\infty} r_x(k)w_B(k)\, e^{-jk\omega} \qquad (3)$$

Where: $w_B(k)$ is a bartlett window given by:

$$w_B(k) = w_R(k) * w_R(-k) = \sum_{n=-\infty}^{\infty} w_R(n-k)w_R(n) \qquad (4)$$

Modified Periodogram Method: Introduction

- From eq(3 & 4), we have :

$$E\left\{\hat{P}_{per}(e^{j\omega})\right\} = \frac{1}{N}\sum_{k=-\infty}^{\infty} r_x(k)w_B(k)\, e^{-jk\omega} = \frac{1}{N}\sum_{k=-\infty}^{\infty} r_x(k)\left[w_R(k) * w_R(-k)\right] e^{-jk\omega} \quad (5)$$

- The right side of eq(5) is the Fourier transform of $r_x(k)\left[w_R(k) * w_R(-k)\right]$, hence:

$$E\left\{\hat{P}_{per}(e^{j\omega})\right\} = \frac{1}{N} DTFT\left\{r_x(k)\left[w_R(k) * w_R(-k)\right]\right\}$$

$$= \frac{1}{N} DTFT\left\{r_x(k)\right\} * \left[\frac{1}{2\pi} DTFT\left\{w_R(k)\right\} DTFT\left\{w_R(-k)\right\}\right]$$

$$= \frac{1}{2\pi N} P_x(e^{j\omega}) * \left[W_R(e^{j\omega})W_R^*(e^{j\omega})\right]$$

$$= \frac{1}{2\pi N} P_x(e^{j\omega}) * \left|W_R(e^{j\omega})\right|^2 \quad (6)$$

Modified Periodogram Method: Introduction

- Where $W_R(e^{j\omega})$ in eq(6) is the Fourier transform of rectangular window given by:

$$W_R(e^{j\omega}) = \frac{\sin\left(N\omega/2\right)}{\sin\left(\omega/2\right)} e^{-j\frac{(N-1)\omega}{2}} \qquad (7)$$

- Smoothing is governed by the choice of window applied to the data
- Although the rectangular window has the narrowest main lobe among all windows, its sidelobes decay relatively slowly
- Other windows can be designed to achieve lower sidelobe levels, at the cost of a wider main lobe:
- **Example:**
    - Bartlett
    - Hamming
    - Hann
    - Blackman

Modified Periodogram Method: Introduction

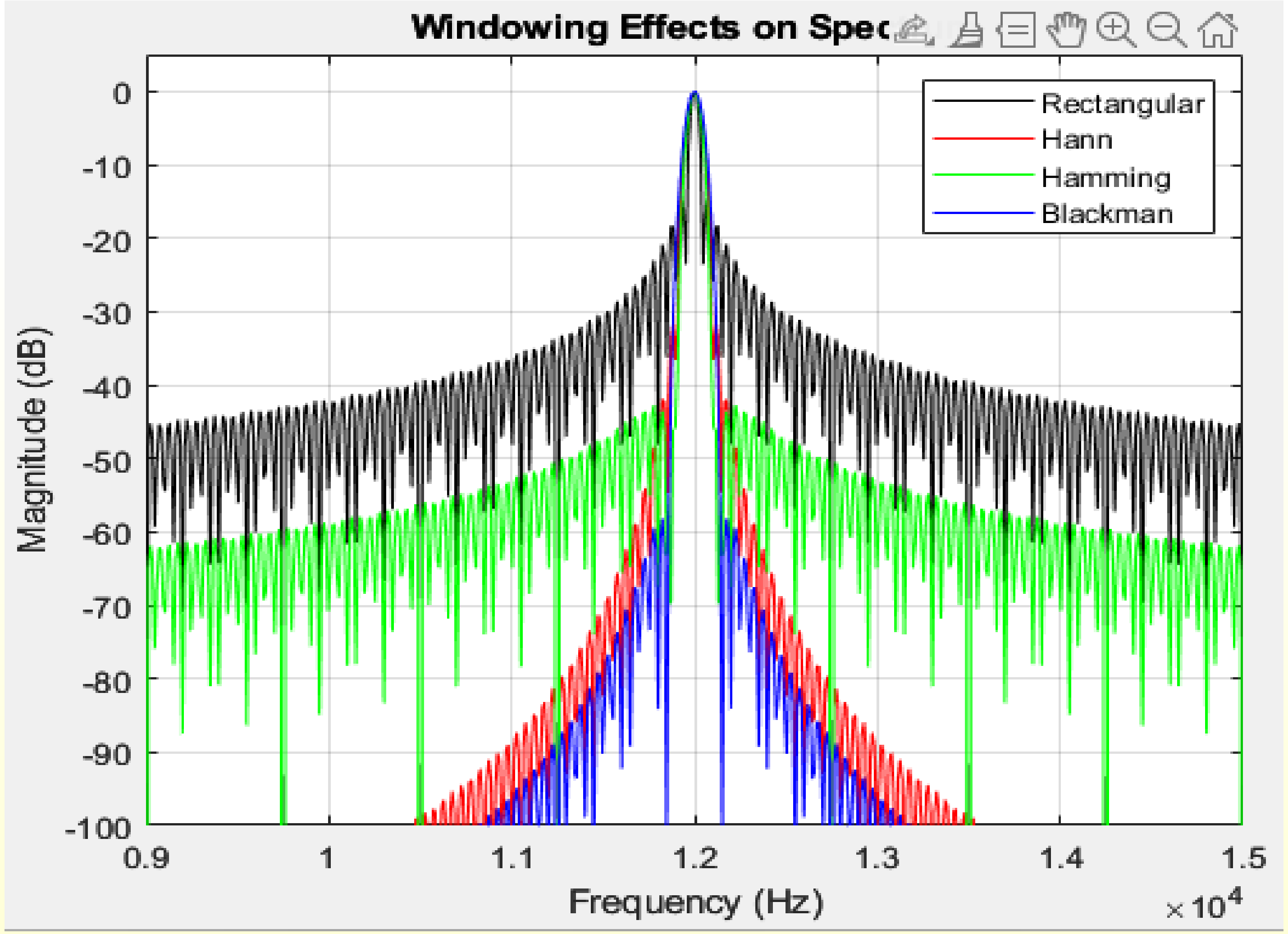
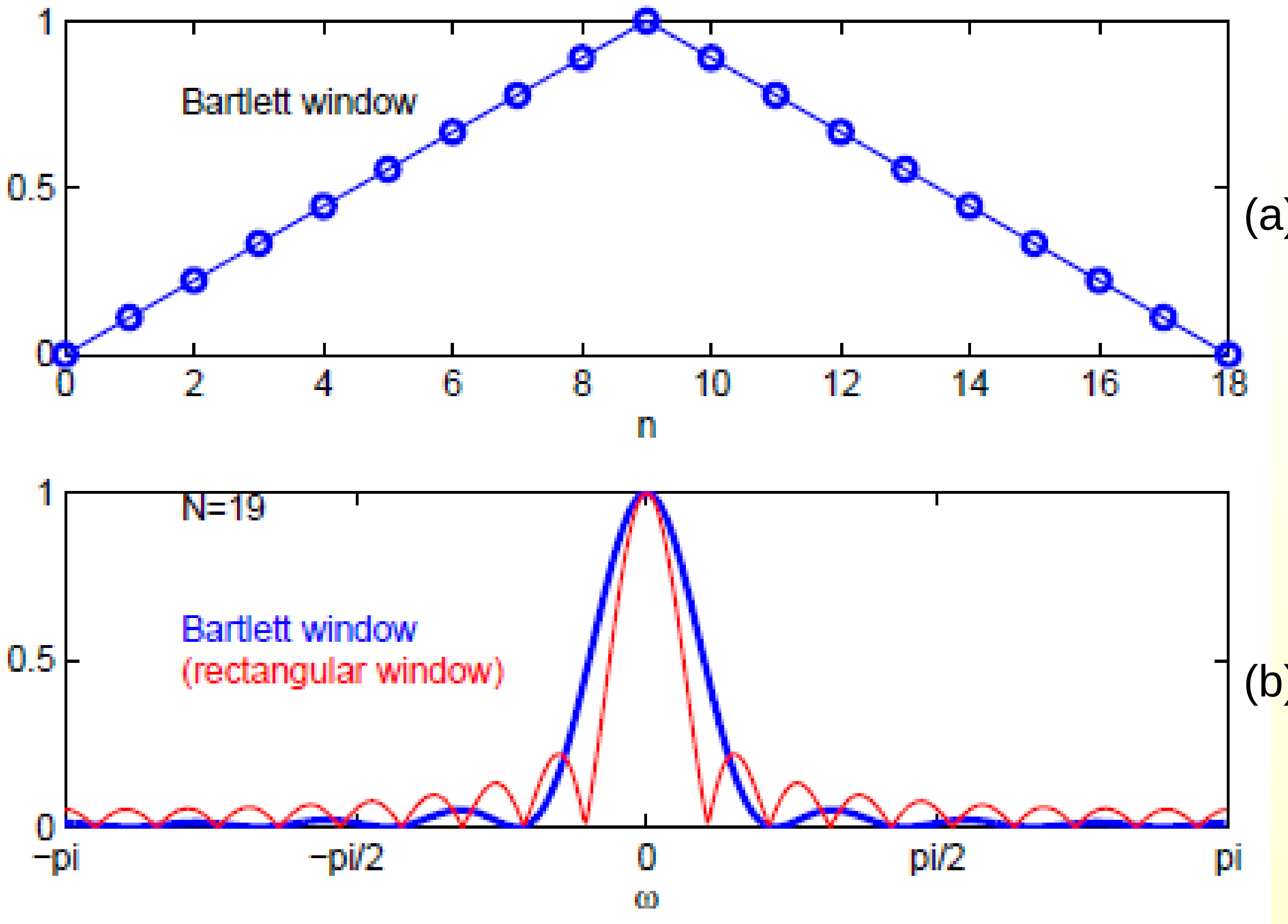


Figure 1: Spectrum of different Windows

Source: "How Windowing Affects Your Periodogram", salimwireless.com. https://blogger.googleusercontent.com/img/b/R29vZ2xl/AVvXsEg3FUd_oLcvERXMplpwwB5QoVwSWZ-j0EDa_GL33UOmVi6HM_Vaj6Z0s3pHSmci5TTs3TsTZcsdArGQJ12Otcw8gIXUZrXMoCs_2zMmTXKlUqMje3s_JhzrihnXUg4NeEAbWfwdoUGyZ6fT3NKtSLGa8pIFVg8j7YsKGV7tflEMP5pYT_B28x5bn4gnb6c/s506/Screenshot%202025-05-17%20181912.png

Modified Periodogram Method: Introduction

- We may also compare the spectrum of rectangular and Bartlett window



(a)

(b)

Figure 2: (a) Bartlett window in time domain for N=19 (b) Bartlett and Rectangular window spectrum for N=19

Source: "Spectrum Estimation", CSDN. https://i-blog.csdnimg.cn/blog_migrate/3106459e35a04af09a5173eb98b35fd5.png#pic_center

Modified Periodogram Method: Introduction

- The spectrum of rectangular Vs Hamming window

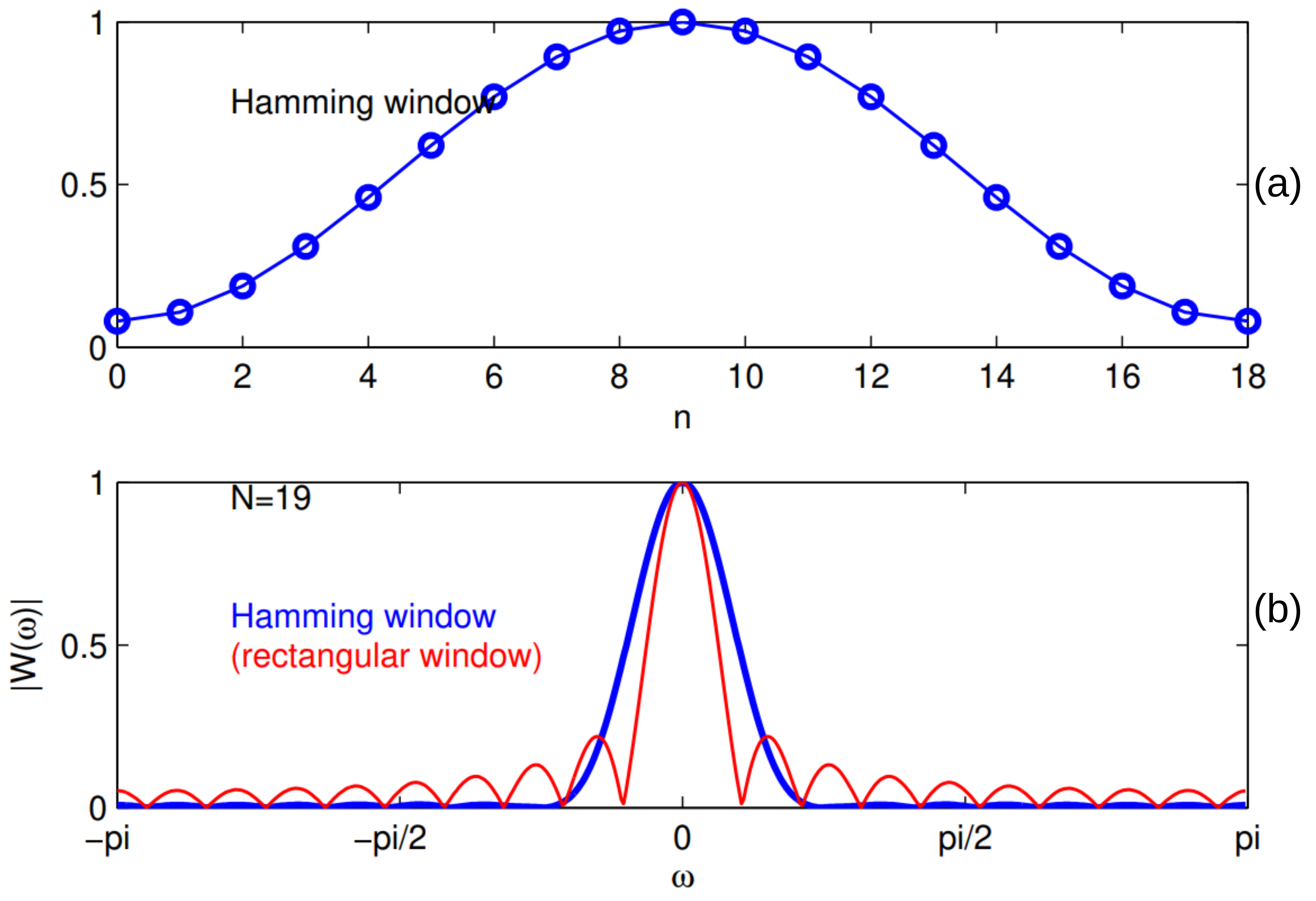


Figure 3: (a) Hamming window in time domain for N=19
(b) Hamming and Rectangular window spectrum for N=19

Source: "Hamming Window Vs Rectangular Window", dedicated-lark. https://custom-images.strikinglycdn.com/res/hrscywv4p/image/upload/c_limit,fl_lossy,h_9000,w_1200,f_auto,q_auto/24944559/628842_594250.png

Modified Periodogram

- Let’s see the periodogram and modified periodogram (using hamming) of $x(n)$ for $N = 128$

$$x(n) = 0.1\sin(0.2\pi n + \phi_1) + \sin(0.3\pi n + \phi_2) + v(n) \qquad (8)$$

(a) (b)

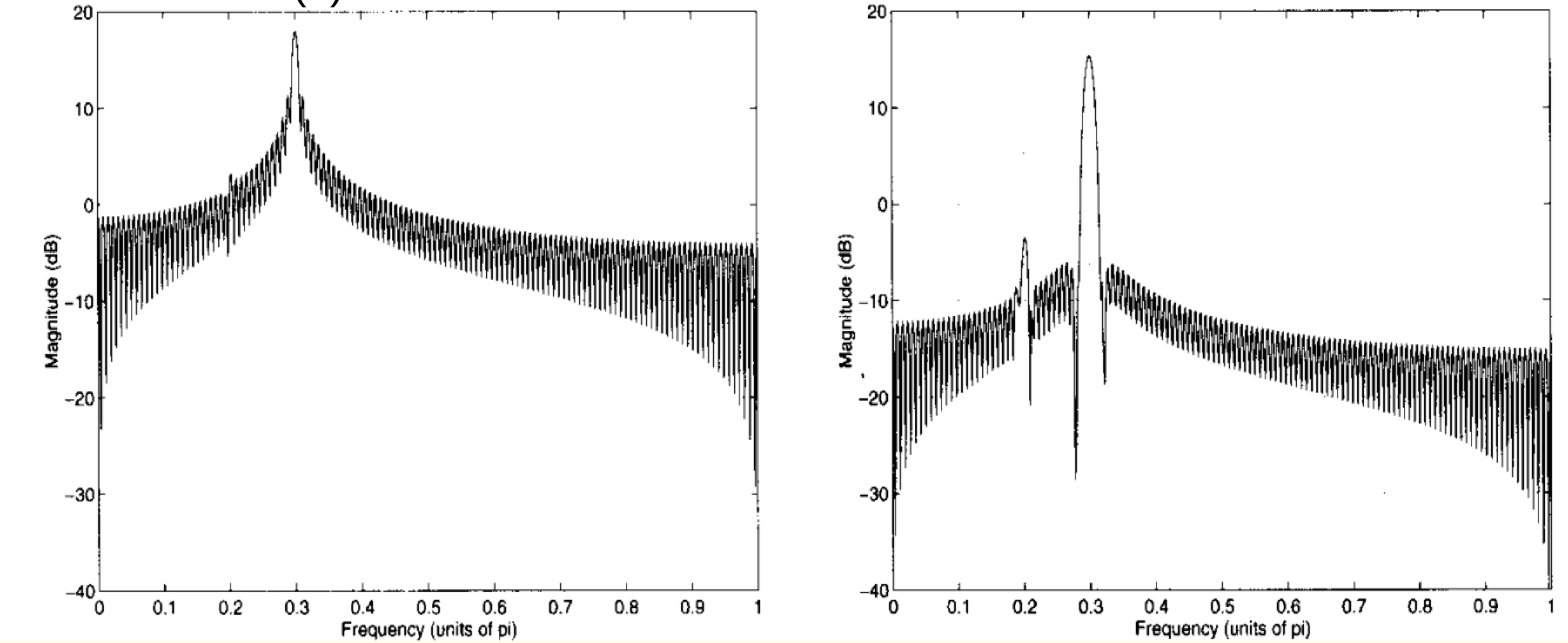


Figure 4: (a) Expected value of the periodogram (b) Expected value of the modified periodogram using Hamming Window

Source: “Periodogram Vs Modified Periodogram using Hamming Window”, dedicated-lark. https://custom-images.strikinglycdn.com/res/hrscywv4p/image/upload/c_limit,fl_lossy,h_9000,w_1200,f_auto,q_auto/24944559/880005_107540.png

Modified Periodogram

- From the previous plot on figure (4a), we can see that the spectral component at frequency of 0.2π is completely masked by the side lobe of the window at frequency 0.3π
- But, for the case of Hamming window, both spectral components are visible since the hamming window has smaller side lobs
- The side lobs of Hamming window are down about by 30dB compared to the side lobs of the rectangular window
- However, there is always a trade-off and the Hamming window exhibits a wider main lobe compared to the rectangular window
- The presence of wide main lobe will also effect the resolution of the Hamming window

Modified Periodogram

- The periodogram of a signal after applying a general window $w(n)$ on the process is referred to as the **Modified Periodogram**
- The modified periodogram of the process for a general window with length N is given by:

$$\hat{P}_M(e^{j\omega}) = \frac{1}{NU}\left|\sum_{n=-\infty}^{\infty} x(n)w(n)e^{-jn\omega}\right|^2 \qquad (9)$$

Where:

$$U = \frac{1}{N}\sum_{n=0}^{N-1}\left|w(n)\right|^2 \qquad (10)$$

- U is a constant calculated from eq(10)

Performance of Modified Periodogram

- Taking the expected value of the modified periodogram to evaluate its bias, we will obtain:

$$E\left\{\hat{P}_M(e^{j\omega})\right\} = \frac{1}{2\pi NU} P_x(e^{j\omega}) * \left|W(e^{j\omega})\right|^2 \qquad (11)$$

Where: $W(e^{j\omega})$ is the Fourier transform of the data window

- It is also noted that U can be written in alternative form using Parzival's Theorem as follows:

$$U = \frac{1}{N}\sum_{n=0}^{N-1}\left|w(n)\right|^2 = \frac{1}{2\pi N}\int_{-\pi}^{\pi}\left|W(e^{j\omega})\right|^2 d\omega \qquad (12)$$

Performance of Modified Periodogram

- Dividing eq(12) both sides by U, we will have :

$$\frac{1}{2\pi NU}\int_{-\pi}^{\pi}\left|W(e^{j\omega})\right|^2 d\omega = 1 \qquad (13)$$

- With a choice of an appropriate window, the term $\left|W(e^{j\omega})\right|^2 / 2\pi NU$ in eq(11) converges to unit impulse as $N \to \infty$
- Hence, the modified periodogram is asymptotically unbiased:

$$\lim_{N\to\infty} E\ \hat{P}_M(e^{j\omega}) = \lim_{N\to\infty}\left[\frac{1}{2\pi NU}P_x(e^{j\omega}) * \left|W(e^{j\omega})\right|^2\right] = P_x(e^{j\omega}) \qquad (14)$$

- It is also noted that if $w(n)$ is a rectangular window, then $U = 1$ and the modified periodogram converges to a periodogram

Performance of Modified Periodogram

- Since Modified periodogram is a periodogram of windowed data sequence, the variance of the modified periodogram is approximately equals to the variance of a periodogram:

$$\textbf{Var}\ \hat{P}_M(e^{j\omega}) \approx P_x^2(e^{j\omega}) \qquad (15)$$

- Hence, Modified Periodogram is not consistent estimate of the power spectrum
- In terms of variance, there is no any change compared to periodogram
- However, the modified periodogram provides a trade of between:
  - Spectral Resolution or Main Lobe Width and,
  - Spectral Masking or Side Lobe Amplitude
- The resolution of modified periodogram can be given in terms of 3dB bandwidth of the main lob as:

$$\textbf{Res}\ \hat{P}_M(e^{j\omega}) \approx (\Delta\omega)_{3dB} \qquad (16)$$

Performance of Modified Periodogram

- The properties of different periodogram is summarized as follows in table below:

Table: Spectral characteristics of different windows [1]

Window Type	Side Lob Amplitude (Db)	3dB Bandwidth $\rightarrow (\Delta\omega)_{3dB}$
Rectangular	-13	$0.89(2\pi/N)$
Bartlett	-27	$1.28(2\pi/N)$
Hanning	-32	$1.44(2\pi/N)$
Hamming	-43	$1.30(2\pi/N)$
Blackman	-58	$1.68(2\pi/N)$

- As shown in the table, using Hamming window reduce the side lobe level by 30dB compared to the rectangular window
- However, There is reduction of spectral resolution for the case of Hamming window compared to the rectangular window

Bartlett's method

- As we have seen previously, both periodogram and modified periodogram are not consistent estimate
- The Bartlett's method uses periodogram averaging approach to produce consistent estimate of the power spectrum
- The motivation behind the Bartlett's method is the fact that the expected value of the periodogram converges to the true value of the power spectrum as $N \to \infty$

$$\lim_{N \to \infty} E\left\{ \hat{P}_{per}(e^{j\omega}) \right\} = P_x(e^{j\omega}) \qquad (17)$$

- Then, If we are able to find the consistent estimate of the mean, $E\left\{ \hat{P}_{per}(e^{j\omega}) \right\}$ then the estimate will be consistent estimate of $P_x(e^{j\omega})$
- Averaging a set of uncorrelated measurements of a random variable yields a consistent estimate of its mean

Bartlett's method

- Hence, Variance of the sample mean will be inversely proportional to the number of measurements, N, and as $N \rightarrow \infty$, the variance goes to zero
- Therefore, we will focus on estimating the power spectrum by averaging multiple programs generated from multiple uncorrelated realizations of the random process
- Let we have the following uncorrelated realizations of the random process:

$$x_i(n) \;\; ; \; for \;\; i = 1, 2, \ldots, K \;\; and \;\; n = 0, 1, \ldots, L-1 \qquad (18)$$

- The periodogram $\hat{P}_{per}^{(i)}(e^{j\omega})$ from $x_i(n)$ can be given by:

$$\hat{P}_{per}^{(i)}(e^{j\omega}) = \frac{1}{L}\left|\sum_{n=0}^{L-1} x_i(n)e^{-jn\omega}\right|^2 \;\; ; \; i = 1, 2, \ldots, K \qquad (19)$$

- Eq(19) revealed that we have K numbers of Periodograms

Bartlett's method

- Averaging those K numbers of periodograms:

$$\hat{P}_x(e^{j\omega}) = \frac{1}{K}\sum_{i=1}^{K}\hat{P}_{per}^{(i)}(e^{j\omega}) \qquad (20)$$

- The expected value of $\hat{P}_x(e^{j\omega})$ is given by:

$$E\left\{\hat{P}_x(e^{j\omega})\right\} = E\left\{\frac{1}{K}\sum_{i=1}^{K}\hat{P}_{per}^{(i)}(e^{j\omega})\right\} = \frac{1}{K}\sum_{i=1}^{K}E\left\{\hat{P}_{per}^{(i)}(e^{j\omega})\right\}$$

$$= \frac{1}{K}\sum_{i=1}^{K}\frac{1}{2\pi}P_x(e^{j\omega}) * W_B(e^{j\omega}) = \frac{1}{K}\left[\frac{K}{2\pi}P_x(e^{j\omega}) * W_B(e^{j\omega})\right]$$

$$= \frac{1}{2\pi}P_x(e^{j\omega}) * W_B(e^{j\omega}) \qquad (21)$$

Bartlett's method

- The $W_B(e^{j\omega})$ in eq(21) the Fourier transform of the Bartlett window $w_B(k)$
- Then similar to periodogram method:

$$\lim_{N\to\infty} E\ \hat{P}_x(e^{j\omega}) = \lim_{N\to\infty}\left[\frac{1}{2\pi}P_x(e^{j\omega}) * W_B(e^{j\omega})\right]$$

$$= P_x(e^{j\omega}) \qquad (22)$$

- Hence $\hat{P}_x(e^{j\omega})$ is an asymptotically unbiased estimate of the power spectrum
- Recalling the realizations or data records are assumed to be uncorrelated, the variance of $\hat{P}_x(e^{j\omega})$ becomes:

$$\mathbf{Var}\ \hat{P}_x(e^{j\omega}) = \frac{1}{K}\mathbf{Var}\ \hat{P}_{per}^{(i)}(e^{j\omega}) \approx \frac{1}{K}P_x^2(e^{j\omega}) \qquad (23)$$

Bartlett's method

- From eq(23), it is noted that the variance of $\hat{P}_x(e^{j\omega})$ goes to zero as $K \to \infty$
- Hence $\hat{P}_x(e^{j\omega})$ is a consistent estimate of the power spectrum provided that both L and K are allowed to go to infinity
- However, the difficulty of the Bartlett's method arise from the absence of uncorrelated realizations since we typically have single realization with length N
- To solve this challenge, Bartlett proposed that partitioning $x(n)$ in to K non overlapping sequence of length L
- Hence, N can be given by:

$$N = KL \qquad (24)$$

- Then, the Bartlett estimate can be computed using eq(19 & 20)

Bartlett's method

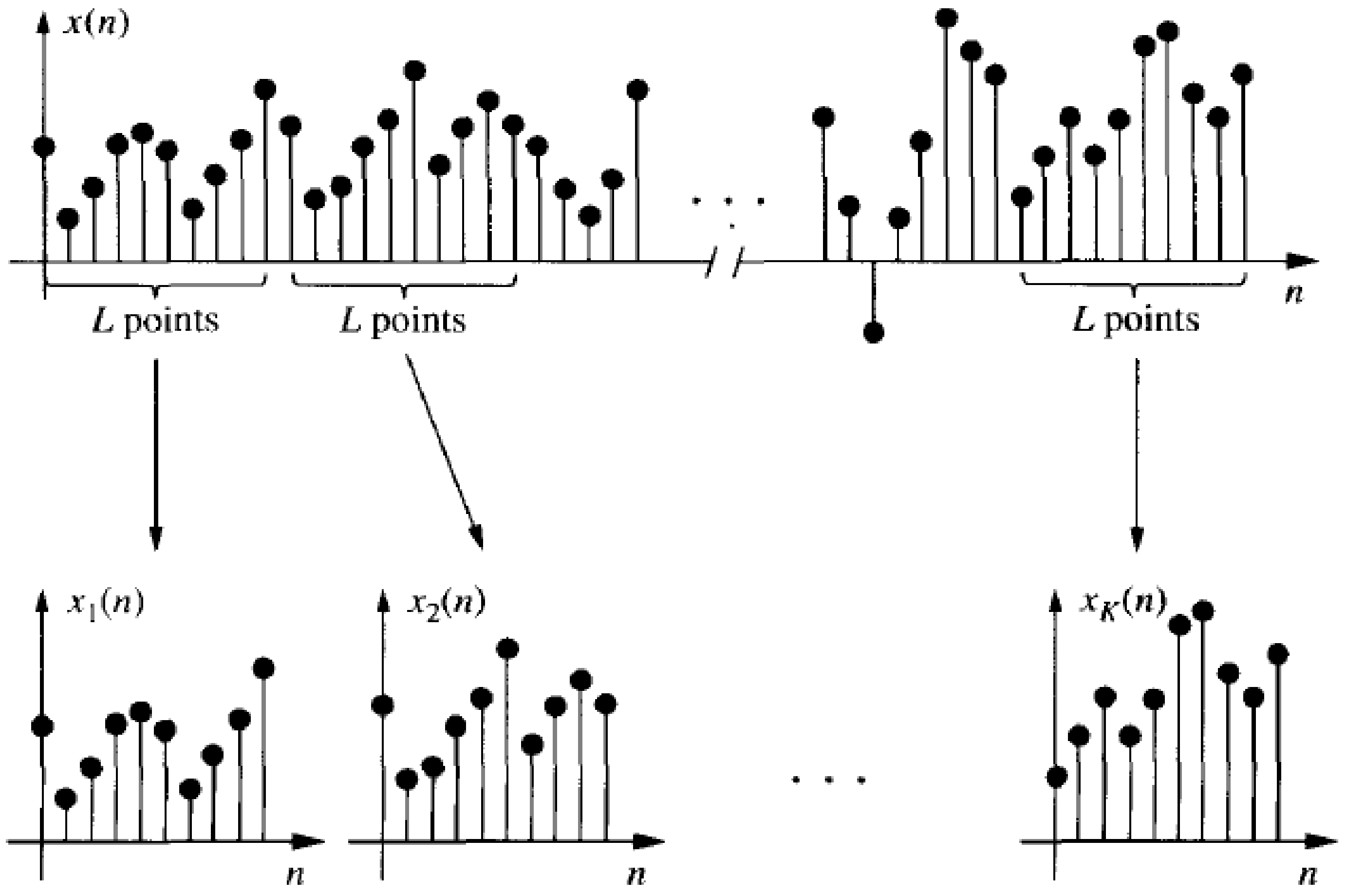


Figure 8.12 *Partitioning x(n) into nonoverlapping subsequences.*

Figure 5: Partitioning the signal data in to non-overlapping sequences

Source: "Spectrum Estimation", CSDN. https://i-blog.csdnimg.cn/blog_migrate/424aefd62ecd032edfaec6780cbd4faf.png#pic_center

Bartlett's method

- Now the partitions, $x_i(n)$, can be written as:

$$x_i(n) = x(n+iL)\ ;\ n = 0,1,2,\ldots,L-1 \quad \textbf{and}\ i = 0,1,2,\ldots,K-1 \tag{25}$$

- Now the Bartlett estimate, $\hat{P}_B(e^{j\omega})$, can be written as:

$$\begin{aligned}
\hat{P}_B(e^{j\omega}) &= \frac{1}{K}\sum_{i=0}^{K-1}\frac{1}{L}\left|\sum_{n=0}^{L-1} x(n+iL)e^{-jn\omega}\right|^2 \\
&= \frac{1}{KL}\sum_{i=0}^{K-1}\left|\sum_{n=0}^{L-1} x(n+iL)e^{-jn\omega}\right|^2 \\
&= \frac{1}{N}\sum_{i=0}^{K-1}\left|\sum_{n=0}^{L-1} x(n+iL)e^{-jn\omega}\right|^2
\end{aligned} \tag{26}$$

Performance of Bartlett's method

- Similarly, the expected value of $\hat{P}_B(e^{j\omega})$ is given by:

$$E\left[\hat{P}_B(e^{j\omega})\right] = \frac{1}{2\pi} P_x(e^{j\omega}) * W_B(e^{j\omega}) \qquad (27)$$

- Thus, the Bartlett estimate is asymptotically unbiased
- Considering the data sequence are approximately uncorrelated, the variance of $\hat{P}_B(e^{j\omega})$ is given by:

$$\mathbf{Var}\left[\hat{P}_B(e^{j\omega})\right] \approx \frac{1}{K} \mathbf{Var}\left[\hat{P}_{per}^{(i)}(e^{j\omega})\right]$$

$$\approx \frac{1}{K} P_x^2(e^{j\omega}) \qquad (28)$$

- Again , if the L and K are allowed to go to infinity, the Bartlett estimate is consistent

Performance of Bartlett's method

- The resolution of $\hat{P}_B(e^{j\omega})$ depends on the length, L , since the periodograms used for the Bartlett estimate computations are computed from sequences of length L:

$$\textbf{Res}\ \ \hat{P}_B(e^{j\omega}) = 0.89\frac{2\pi}{L}$$
$$= 0.89K\frac{2\pi}{N} \qquad (30)$$

- Hence , its resolution is K times worth than the periodogram
- We can see also that the is a trade off between variance and spectral resolution
- When K increases the variance will decrease but the resolution will be degraded since L will be also reduced
- On contrary, the resolution will be improved and variance will be increased when K decreases since L will also be increased

Performance of Bartlett's method

- Bartlett estimate of $x(n) = 15\sin(0.22\pi n + \phi) + 10\sin(0.24\pi n + \phi)$

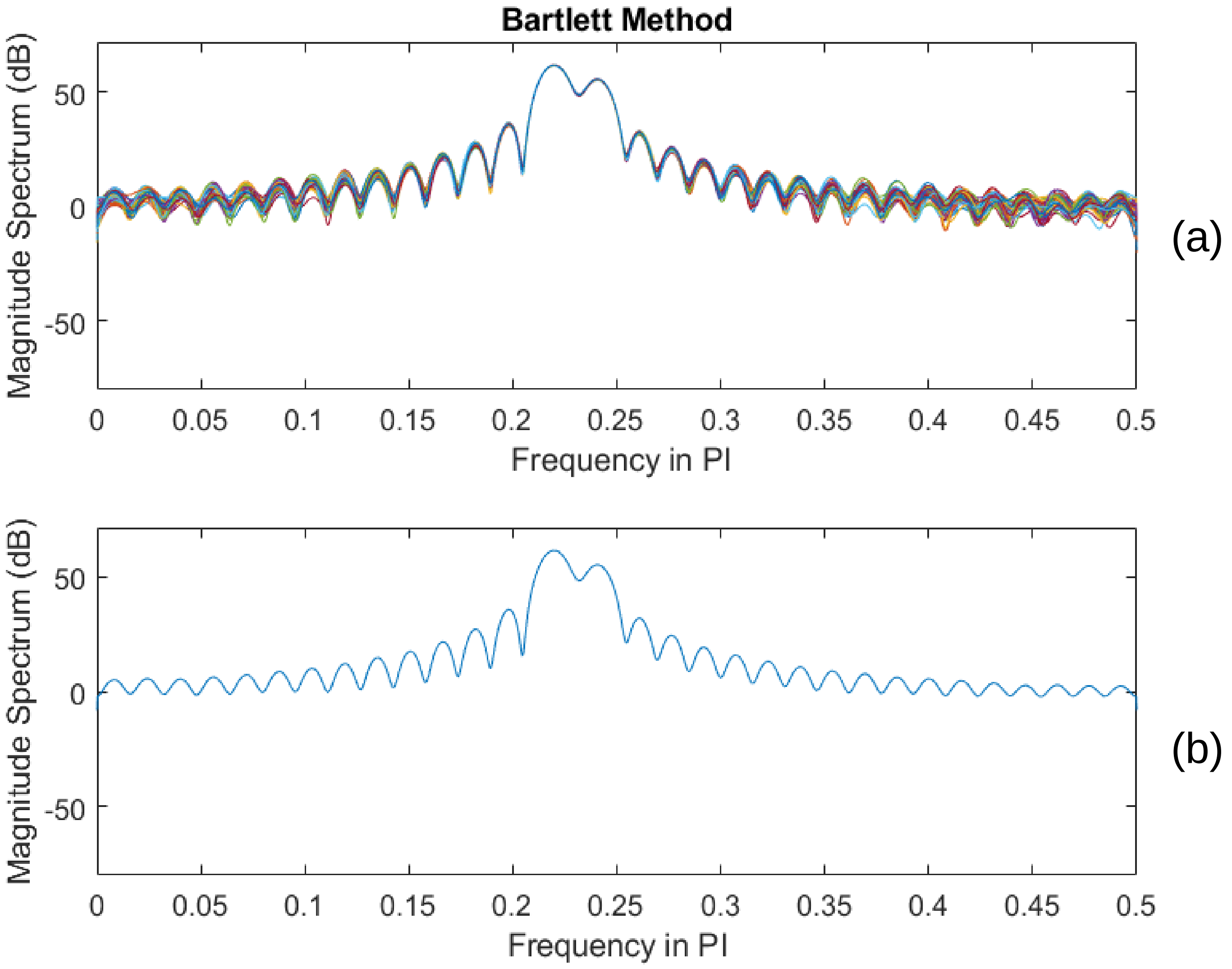


Figure 6: Bartlett estimate of two sinusoids (a) Overlay plot of 50 Bartlett estimate using N=512 and K=8 (b) the ensemble average

Source: MATLAB-generated plot using modified source code adapted from: S. Lakshmi, *"Spectrum-estimation,"* GitHub repository. https://github.com/srilakshmialla/Spectrum-estimation

Performance of Bartlett’s method

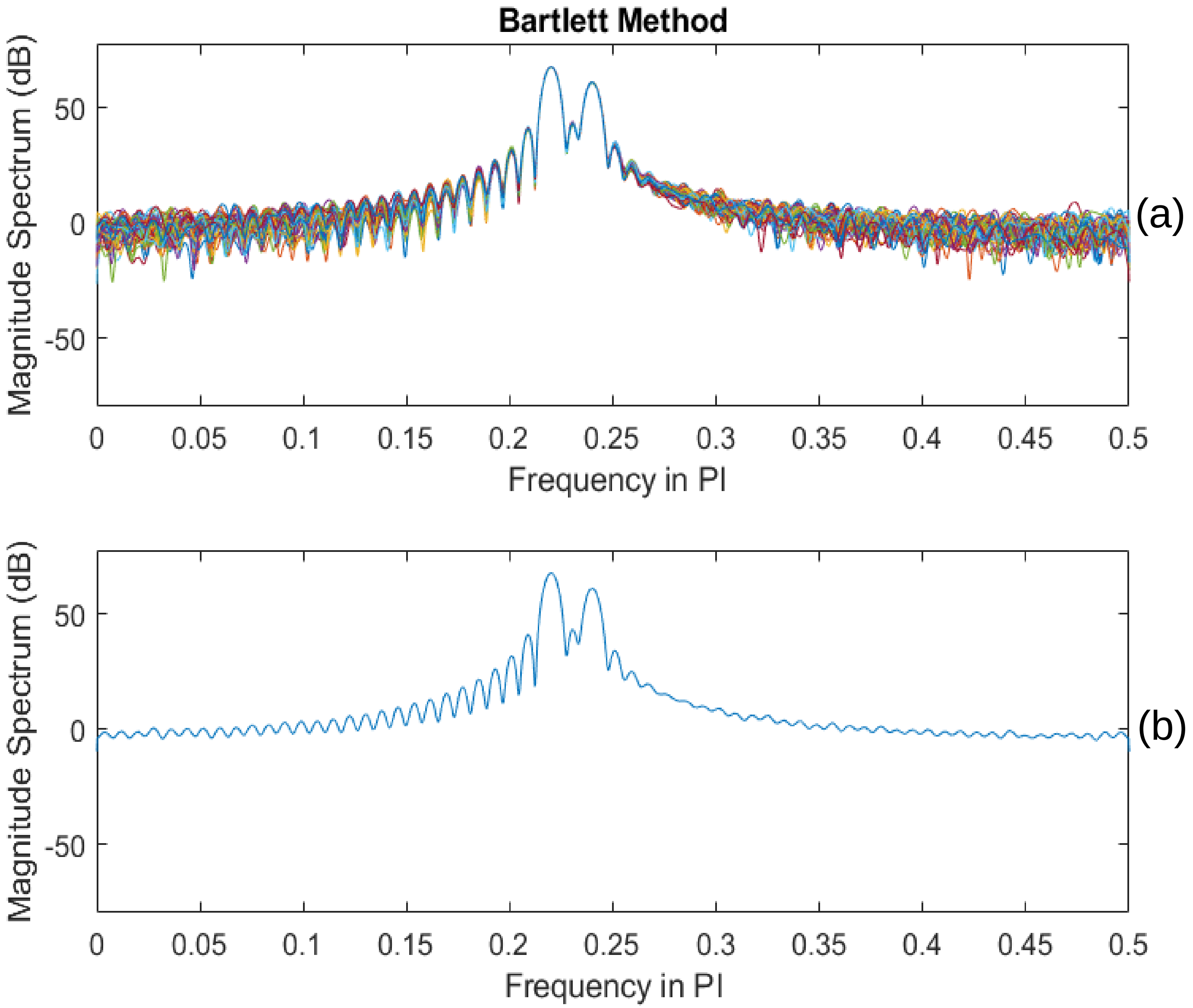


Figure 7: Bartlett estimate of two sinusoids
(a) Overlay plot of 50 Bartlett estimate using N=512 and K=4
(b) the ensemble average

Source: MATLAB-generated plot using modified source code adapted from: S. Lakshmi, *“Spectrum-estimation,”* GitHub repository. https://github.com/srilakshmialla/Spectrum-estimation

Performance of Bartlett's method

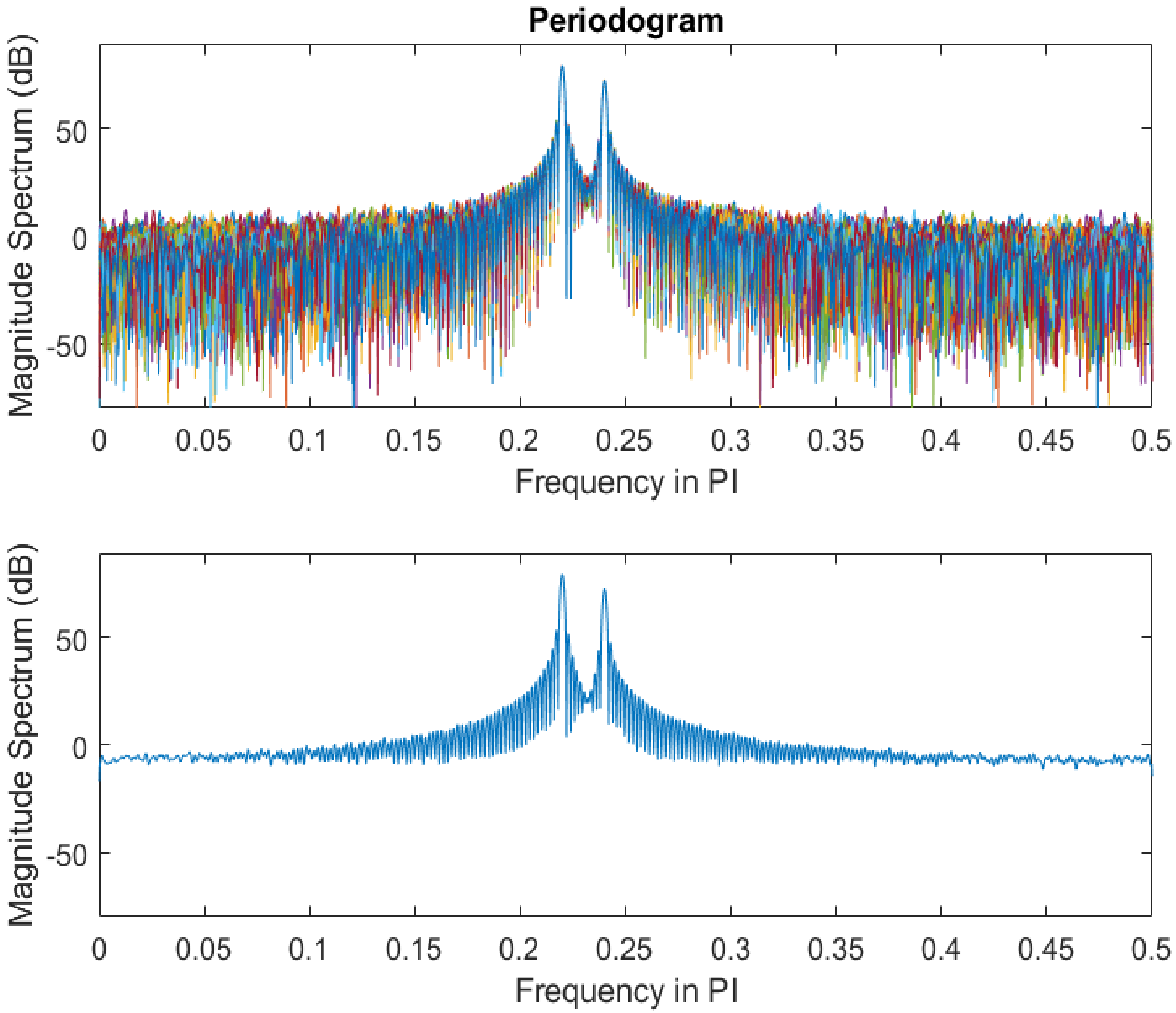


Figure 8: Periodogram of two sinusoids (a) Overlay plot of 50 periodograms using N=512 (b) the periodogram average

Source: MATLAB-generated plot using modified source code adapted from: S. Lakshmi, *"Spectrum-estimation,"* GitHub repository. https://github.com/srilakshmialla/Spectrum-estimation

Performance of Bartlett’s method

- Variance of Bartlett estimate for unit variance white noise for different values of K

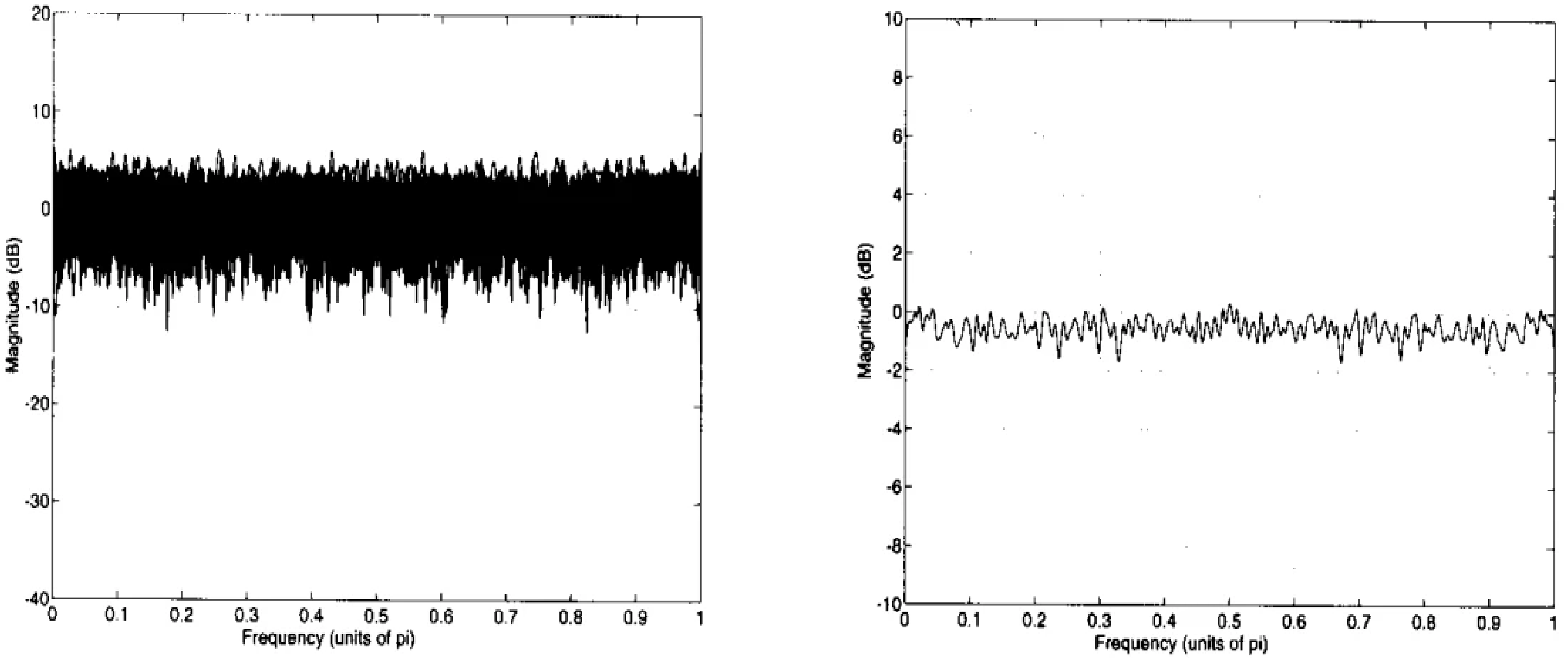


Figure 9: (a) Overlay plot of 50 Bartlett estimate using N=512 and K=4 (b) the ensemble average

Source: *“Non Parametric Spectrum-estimation,”* gray-sparrow. https://custom-images.strikinglycdn.com/res/hrscywv4p/image/upload/c_limit,fl_lossy,h_9000,w_1200,f_auto,q_auto/24525391/261389_721060.png

Performance of Bartlett's method

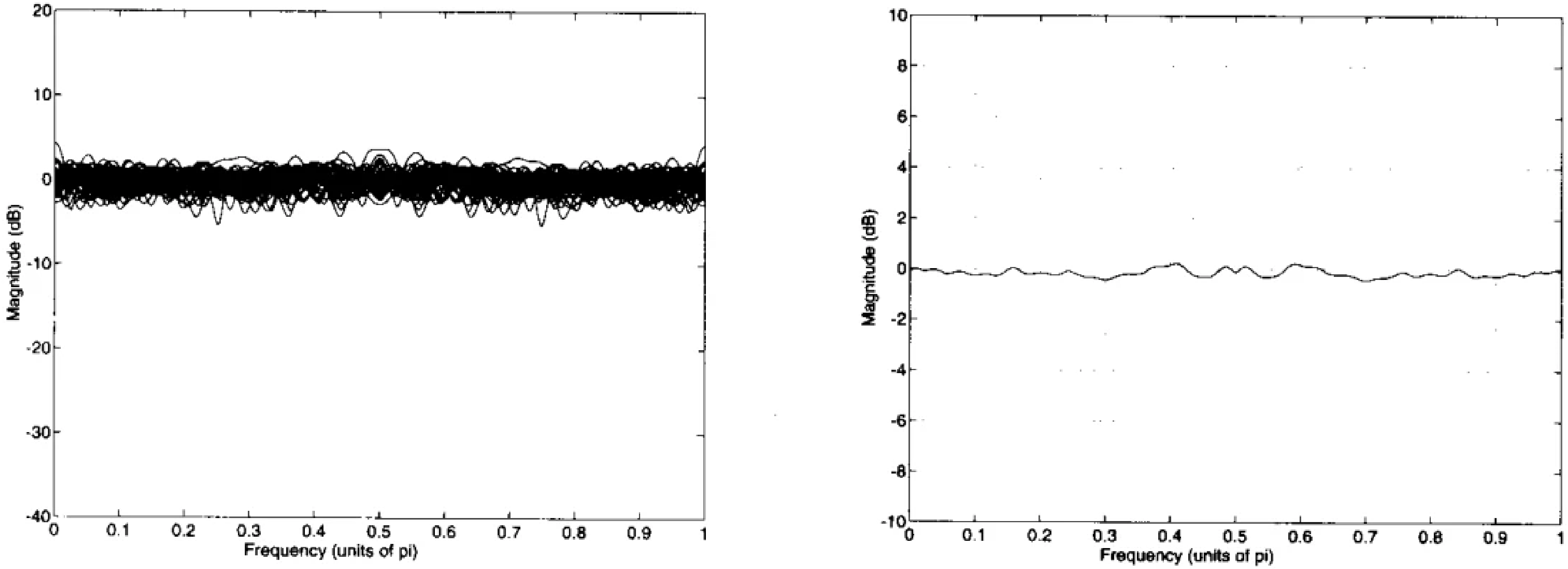


Figure 10: (a) Overlay plot of 50 Bartlett estimate using N=512 and K=8 (b) the ensemble average

Source: *"Non Parametric Spectrum-estimation,"* gray-sparrow. https://custom-images.strikinglycdn.com/res/hrscywv4p/image/upload/c_limit,fl_lossy,h_9000,w_1200,f_auto,q_auto/24525391/580785_410368.png

Performance of Bartlett’s method

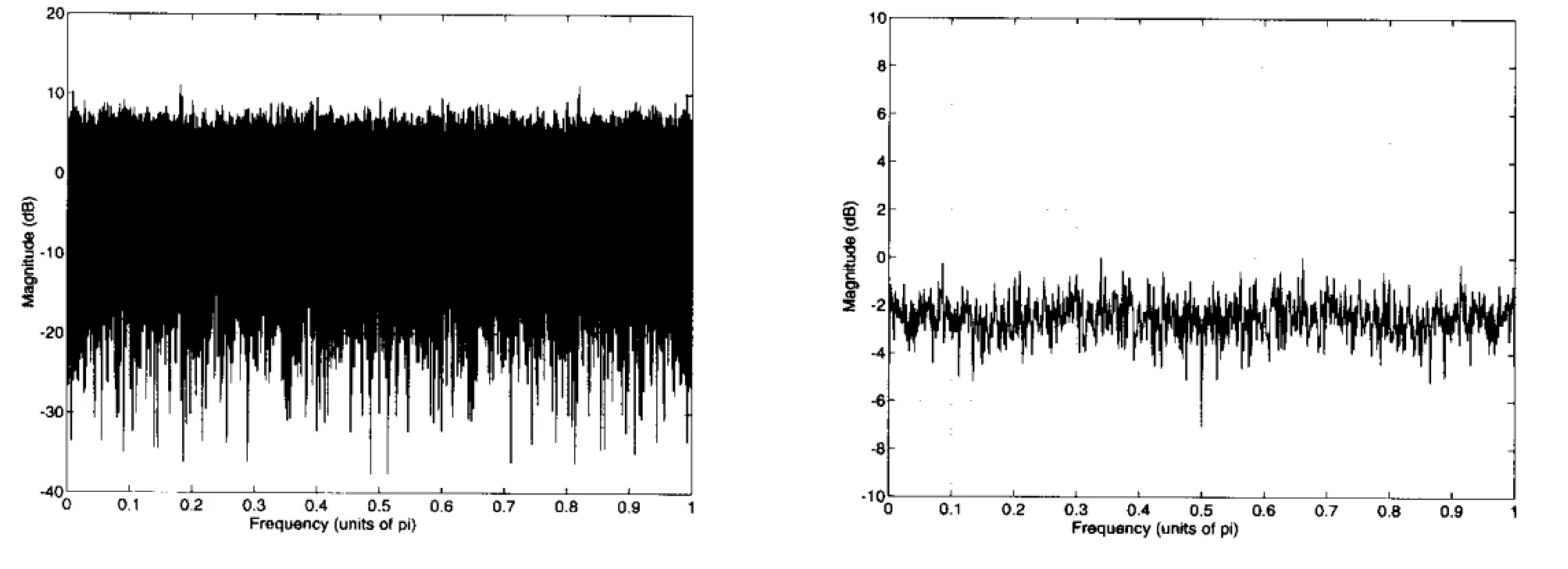


Figure 11: (a) Overlay plot of 50 Periodogram using N=512 (b) the ensemble average

Source: *“Non Parametric Spectrum-estimation,”* gray-sparrow. https://custom-images.strikinglycdn.com/res/hrscywv4p/image/upload/c_limit,fl_lossy,h_9000,w_1200,f_auto,q_auto/24525391/94970_292149.png

Summary

- **Modified Periodogram Method:**
  - ✓ **Motivation:** Due to the choice of rectangular window, the performance of the periodogram is affected by high spectral masking by the side lobs due to the high power leakage to the side lobs
  - ✓ Therefore, to reduce the side lob spectral masking modified periodogram utilize other type of window function such as: Bartlett, Hann, Hamming, and Blackman windows
  - ✓ However, the resolution of the periodogram is better than the modified program since the rectangular window posses narrow main lob compared to others
- **Performance of Modified Periodogram**
  - ✓ Asymptotically Unbiased estimator of the power spectrum
  - ✓ Non Consistent estimator of the power spectrum

Summary

- **Bartlett Method:**
  - **Motivation:**
    - The periodogram and the modified periodogram are not consistent estimate of the true power spectrum
    - However, Variance of the sample mean will be inversely proportional to the number of measurements and the variance goes to zero for large data samples
  - Therefore, the Bartlett method uses periodogram average by averaging preprograms which are constructed from uncorrelated realizations
  - Uses data partitioning to tackle uncorrelated measurement unavailability
- **Performance of Bartlett Estimate**
  - Asymptotically Unbiased estimator of the power spectrum
  - Consistent estimator of the power spectrum

References

[1] Monson H. Hayes, “Statistical Digital Signal Processing and Modeling’’, John Wiley and sons, Pp.411, 1996.

Contents Here

Thank You!