Statistical Digital Signal Processing

Week 11 Non- Parametric Spectrum Estimation: Welch and Blackman Tukey Methods

Lecturer: ***Zelalem Hailu (Assistant Prof.)***

Addis Ababa Science and Technology University,
Addis Ababa, Ethiopia

Previous Topic (Week-10)

Modified Periodogram and Bartlett Methods

- Modified Periodogram
- Performance of Modified Periodogram
- Bartlett's Method
- Performance of Bartlett's Method

Lecture Learning Outcomes

1. Explain the principles and procedures of the Welch Method for spectral estimation
2. Apply the Welch Method to analyze signals and estimate power spectral density
3. Evaluate the performance of the Welch Method in terms of bias and variance
4. Describe the Blackman–Tukey Method and its role in spectral analysis
5. Compare the performance of the Welch and Blackman–Tukey Methods with respect to bias, variance, and estimation accuracy

Week 11: Welch and Blackman Tukey methods

Outline

- Welch Method
- Performance of Welch Method (Bias and Variance)
- Blackman Tukey method
- Performance of Blackman Tukey Method (Bias and Variance)

Welch Methods

- Welch Method has made two modifications to Bartlett method which are:
  - Allow the sequence $x_i(n)$ to overlap
  - Perform Modified Periodogram averaging rather than Periodogram averaging
- Let's assume successive sequence having data length L per sequence are offset by D points
- Then, the **ith** sequence is given by:

$$x_i(n) = x(n+iD) \;\; ; \;\; n = 0,1,2\ldots,L-1 \qquad (1)$$

- Hence, the amount of overlap between $x_i(n)$ and $x_{i+1}(n)$ is L-D
- If the entire available N data are covered with K sequences, N, L, and D can be related by the following equation:

$$N = L + D(K-1) \qquad (2)$$

Welch Methods

- **Note:** If there is no overlap, D is equal to L, and we have:

$$K = \frac{N}{L} \qquad (3)$$

Similar to Bartlett Method

- For **50%** overlap, we will have:

$$D = \frac{L}{2} \qquad (4)$$

$$K = 2\frac{N}{L} - 1 \qquad (5)$$

Welch Methods

- If we maintain the same resolution as Bartlett's method by maintaining the same section length L, we will have approximately double modified periodogram for Welch with 50% overlap compared to Bartlett method as rivaled by eq(5)
- For similar number of sequence, K, for both Bartlett and Welch, we have:

$$K = \frac{N}{L_{Welch}} - 1 = \frac{N}{2L} - 1 \qquad (6)$$

Where:

L is the sequence length for Bartlett

L_{Welch} is the sequence length for Welch

- Hence, for similar achieved variance, Welch method provides better resolution compared to the Bartlett method

Performance of Welch Methods: Bias

- The Welch estimate can be written by averaging the modified Periodograms as:

$$\hat{P}_W(e^{j\omega}) = \frac{1}{K}\sum_{i=0}^{K-1}\hat{P}_M^{(i)}(e^{j\omega}) \qquad (7)$$

Where:

$$\hat{P}_M^{(i)}(e^{j\omega}) = \frac{1}{LU}\left|\sum_{n=0}^{L-1} w(n)x(n+iD)e^{-jn\omega}\right|^2 \;;\; i = 0,1,\ldots,K-1 \qquad (8)$$

- From eq(7 & 8), the Welch estimate can be written as:

$$\hat{P}_M^{(i)}(e^{j\omega}) = \frac{1}{LUK}\sum_{i=0}^{K-1}\left|\sum_{n=0}^{L-1} w(n)x(n+iD)e^{-jn\omega}\right|^2 \qquad (9)$$

Performance of Welch Methods: Bias

- Evaluating the expected value of Welch estimate to check its bias, we will have:

$$E\left\{\hat{P}_W(e^{j\omega})\right\} = \frac{1}{K}\sum_{i=0}^{K-1} E\left\{\hat{P}_M^{(i)}(e^{j\omega})\right\} = P_x(e^{j\omega}) * \frac{1}{2\pi LU}\left|W(e^{j\omega})\right|^2 \qquad (10)$$

Where:

$W(e^{j\omega})$ is the Fourier transform of L data points $w(n)$ used to construct modified Periodogram

- Therefore, Welch method is an asymptotically unbiased similar to the previous Periodogram based methods
- The resolution is however depend on the choice of data window as follows [1]

Window Type	3dB Bandwidth
Bartlett	$1.28(2\pi/N)$
Hann	$1.44(2\pi/N)$
Hamming	$1.30(2\pi/N)$
Blackman	$1.68(2\pi/N)$

Performance of Welch Methods: Variance

- The variance of Welch method is difficult to compute since the modified periodogram can not be assumed uncorrelated with the presence of overlap
- Using Bartlett window with 50% overlap, the approximate variance of Welch method given by:

$$Var\ \hat{P}_W(e^{j\omega}) \approx \frac{9}{8K} P_x^2(e^{j\omega}) \qquad (11)$$

- Compared to the Bartlett estimate variance $P_x^2(e^{j\omega})/K$ for a given number of sequence, K, the Welch method variance is larger by the factor of 9/8
- However, for fixed number of entire data point, N, and sequence length, L, with 50% of overlap twice as many section can be averaged for Welch method compared to Bartlett Method

Performance of Welch Methods: Variance

- Assuming 50% of overlap, the variance can be written in terms of L and N as:

$$Var\left\{\hat{P}_W(e^{j\omega})\right\} \approx \frac{9}{16}\frac{L}{N}P_x^2(e^{j\omega}) \qquad (12)$$

- Since $K = N/L$ is the number of sequence used in Bartlett's method, eq(12) becomes:

$$Var\left\{\hat{P}_W(e^{j\omega})\right\} \approx \frac{9}{16}\frac{L}{N}P_x^2(e^{j\omega}) = \frac{9}{16}\left[\frac{1}{K}P_x^2(e^{j\omega})\right]$$

$$\approx \frac{9}{16}\left[Var\left\{\hat{P}_B(e^{j\omega})\right\}\right] \qquad (13)$$

- it is also possible to average more sequence for a given amount of data by increasing the amount of overlap
- However, the computational requirement increases in proportion to K

Performance of Welch Methods: Variance

- In addition, when the amount of overlap increased the correlation between sequences also increase and it has its own negative outcome on the performance
- Therefore, the most commonly used typical overlap amount is either 50% or 75%
- Lets compare the Bartlett estimate and the Welch estimate for the following process $x(n)$

$$x(n) = 15\sin(0.2\pi n + \phi_1) + 10\sin(0.22\pi n + \phi_2) + w(n)$$

For the following Parameters

$N = 512$

$K = 4$

50% Overlap for Welch method

$w(n)$ is unit variance white noise

Performance of Welch Methods: Variance

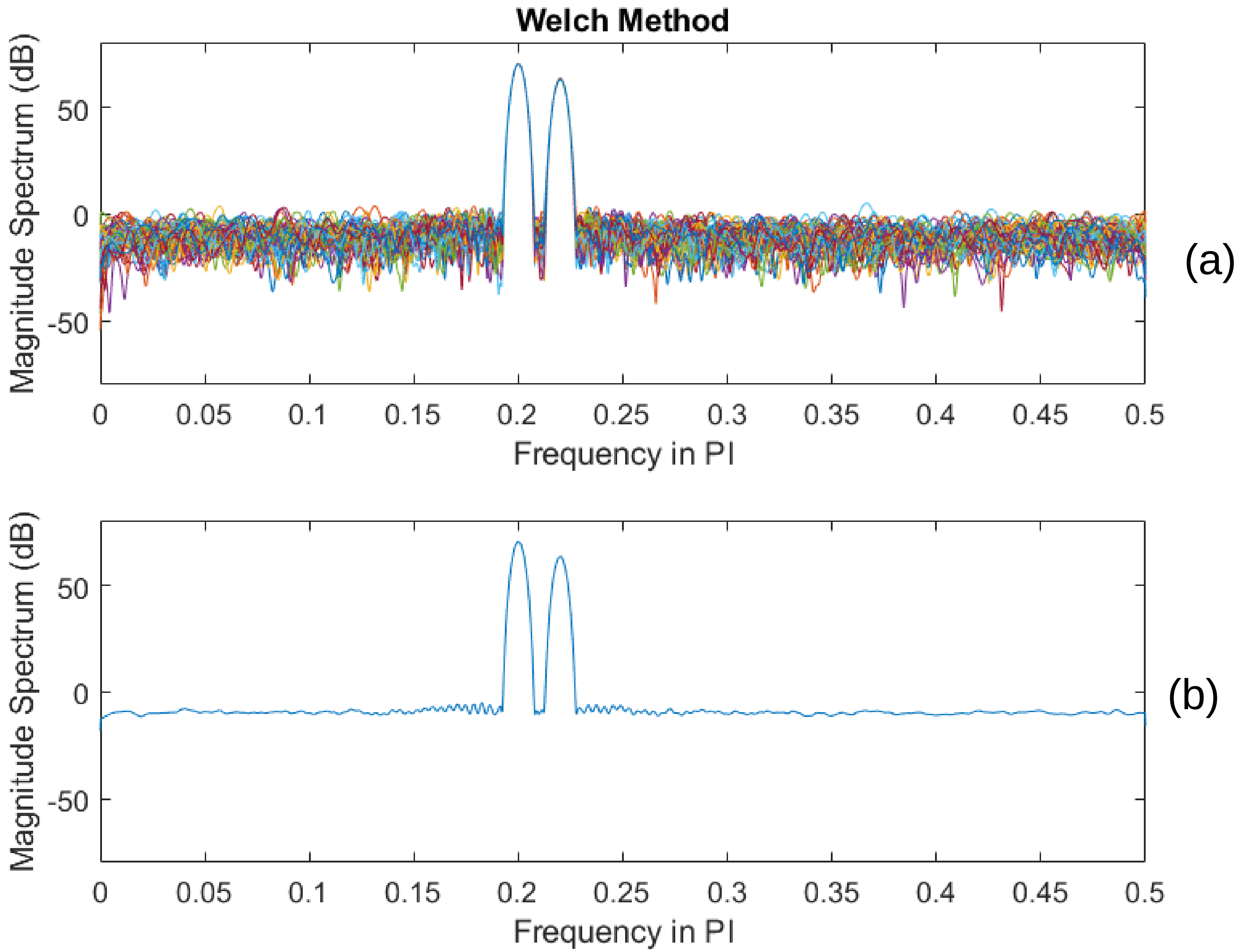

Figure 1: Welch estimate of two sinusoids using Hamming window and 50% overlap (a) Overlay plot of 50 Welch estimate using N=512 and K=4 (b) the ensemble average

Source: MATLAB-generated plot using modified source code adapted from: S. Lakshmi, *"Spectrum-estimation,"* GitHub repository. https://github.com/srilakshmialla/Spectrum-estimation

Performance of Welch Methods: Variance

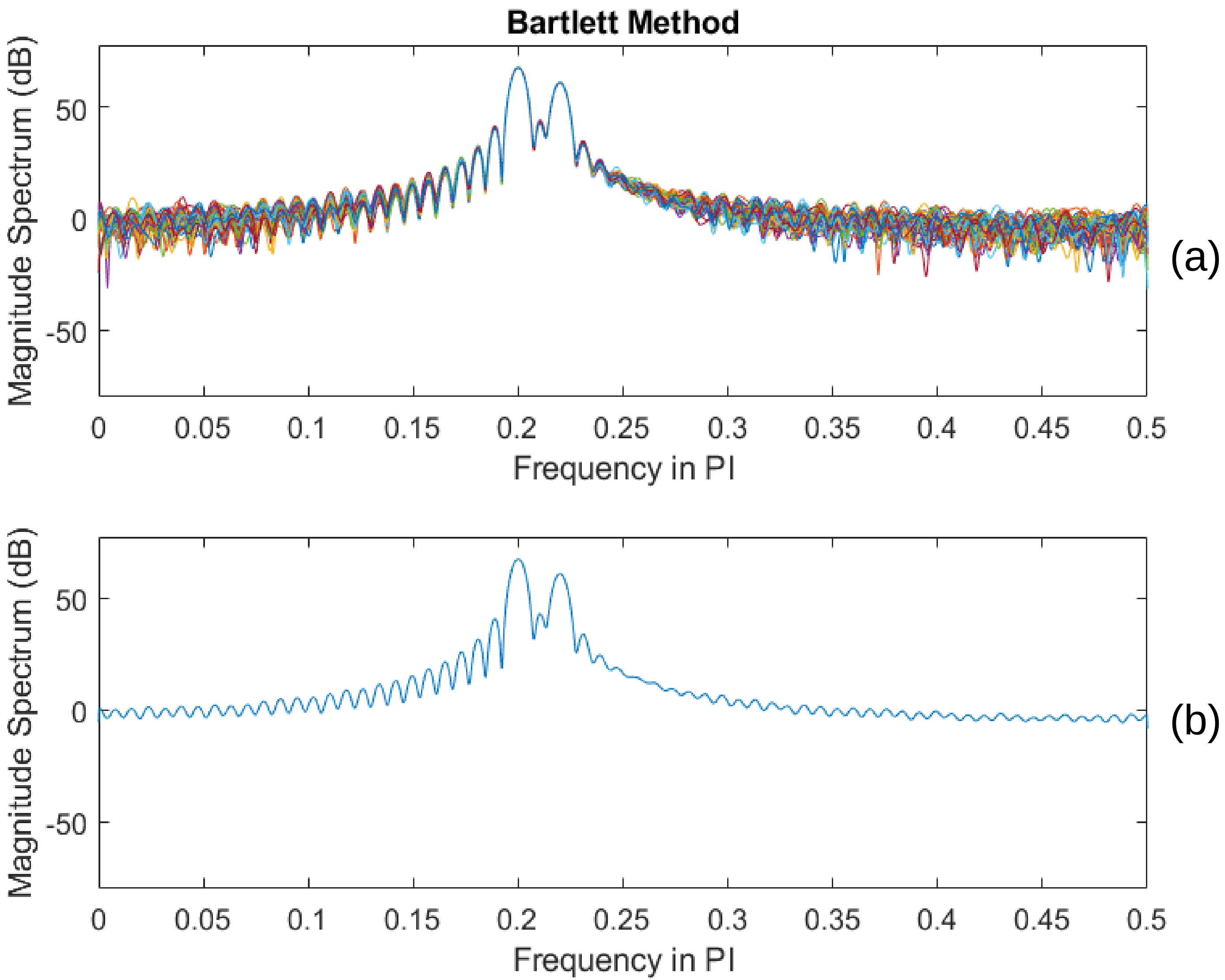

Figure 2: Bartlett estimate of two sinusoids
(a) Overlay plot of 50 Bartlett estimate using N=512 and K=4
(b) the ensemble average

Source: MATLAB-generated plot using modified source code adapted from: S. Lakshmi, *"Spectrum-estimation,"* GitHub repository. https://github.com/srilakshmialla/Spectrum-estimation

Blackman Tukey Method

- The Bartlett and Welch methods aim to reduce the variance of the periodogram by averaging standard and modified periodograms, respectively
- Another approach for reducing the statistical fluctuations of the periodogram is periodogram smoothing, commonly known as the **Blackman–Tukey method**
- Periodogram smoothing reduces variance by applying the Fourier transform to a consistent autocorrelation estimate
- For a finite data record of length N, the estimated variance of $\hat{r}_x(k)$ becomes significantly larger when the lag k approaches N.
- For example $\hat{r}_x(k)$ at lag $k = N - 1$ can be given by:

$$\hat{r}_x(k) = \hat{r}_x(N-1) = \frac{1}{N} x(N-1)x(0) \qquad (14)$$

Blackman Tukey Method

- Because little averaging is involved in estimating $r_x(k)$ for $|k| \approx N$, the estimates remain unreliable regardless of how large N becomes
- Therefore, the variance of the periodogram can only be reduced by either reducing the variance of these estimates or minimizing their contribution for the periodogram
- In Bartlett and Welch methods, the variance of the periodogram is reduced by decreasing the variance of the autocorrelation estimate by averaging
- In the Blackman Tukey method, the variance of the periodogram is reduced by windowing the autocorrelation estimates $\hat{r}_x(k)$, thereby minimizing the effect of unreliable estimates on the spectrum [2]
- The Blackman Tukey spectrum estimate is given by:

$$\hat{P}_{BT}(e^{j\omega}) = \sum_{k=-M}^{M} \hat{r}_x(k) w(k) e^{-jk\omega} \qquad (15)$$

Blackman Tukey Method

- In eq(15), $w(k)$ is the lag window applied to the autocorrelation estimate
- Suppose $w(k)$ is a rectangular window extending from $-M$ to M with $M < N-1$, the estimate of $r_x(k)$ having largest variance will be set to zero
- As a result, the power spectrum estimate will have smaller variance
- However, the is a trade off between the reducing the variance and degrading the resolution since smaller number of autocorrelation estimates are used to form the power spectrum estimate
- As noted from eq(15), the Blackman Tukey spectrum estimate is the Fourier transform of $\hat{r}_x(k)w(k)$
- Hence, we can write $\hat{P}_{BT}(e^{j\omega})$ using frequency convolution theorem as

$$\hat{P}_{BT}(e^{j\omega}) = DTFT\ \ \hat{r}_x(k)w(k) \ = \frac{1}{2\pi}\hat{P}_{per}(e^{j\omega}) * W(e^{j\omega})$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} \hat{P}_{per}(e^{ju})\ W(e^{j(\omega-u)})\ du \qquad (16)$$

Blackman Tukey Method

- Therefore, as rivaled by eq(16), the Blackman Tukey method smoothes the periodogram by convolving with the Fourier transform of the autocorrelation window
- No matter how there is a flexibility in the choice of the window $w(k)$, it has to satisfy the following conditions:
  - $w(k)$ should be conjugate symmetric to make $W(e^{j\omega})$ real valued and;
  - To be guaranteed for having non negative $\hat{P}_{BT}(e^{j\omega})$, $w(k)$ should have non-negative Fourier transform:

$$W(e^{j\omega}) > 0 \qquad (17)$$

- It is noted that the resolution of the Blackman Tukey method depends on the choice of the window type
- However, to evaluate the performance of Blackman Tukey method, it is required to evaluate the bias and variance of the estimate

Performance of Blackman Tukey Method: Bias

- By taking the expected value of eq(16), the bias can be calculated as:

$$E\left\{\hat{P}_{BT}(e^{j\omega})\right\} = E\left\{\frac{1}{2\pi}\hat{P}_{per}(e^{j\omega}) * W(e^{j\omega})\right\}$$

$$= \frac{1}{2\pi} E\left\{\hat{P}_{per}(e^{j\omega})\right\} * W(e^{j\omega}) \qquad (18)$$

- But in the previous lectures we have proved that the expected value of the periodogram is given by :

$$E\left\{\hat{P}_{per}(e^{j\omega})\right\} = \frac{1}{2\pi} P_x(e^{j\omega}) * W_b(e^{j\omega}) \qquad (19)$$

- Then, substituting eq(19) for expected value of the periodogram given in eq(18), we will have:

Performance of Blackman Tukey Method: Bias

$$E\;\;\hat{P}_{BT}(e^{j\omega}) = \frac{1}{2\pi} E\;\;\hat{P}_{per}(e^{j\omega}) * W(e^{j\omega})$$

$$= \frac{1}{2\pi}\left[\frac{1}{2\pi} P_x(e^{j\omega}) * W_b(e^{j\omega})\right] * W(e^{j\omega})$$

$$= \frac{1}{4\pi^2} P_x(e^{j\omega}) * W_b(e^{j\omega}) * W(e^{j\omega}) \qquad (20)$$

- We can write eq(20) in the following equivalent form:

$$E\;\;\hat{P}_{BT}(e^{j\omega}) = \sum_{k=-M}^{M} r_x(k) w_B(k) w(k) e^{-jk\omega} \qquad (21)$$

- If we let:

$$w_{BT} = w_B(k) w(k) \qquad (22)$$

Performance of Blackman Tukey Method: Bias

- We will have:

$$E\left[\hat{P}_{BT}(e^{j\omega})\right] = \sum_{k=-M}^{M} r_x(k) w_B(k) w(k) e^{-jk\omega}$$

$$= \sum_{k=-M}^{M} r_x(k) w_{BT}(k) e^{-jk\omega} = DTFT\left[r_x(k) w_{BT}(k)\right]$$

$$= \frac{1}{2\pi} P_x(e^{j\omega}) * W_{BT}(e^{j\omega}) \qquad (23)$$

- With an assumption of $M \ll N$, we will have

$$w_{BT} = w_B(k) w(k) \approx w(k) \qquad (24)$$

- From eq(23), it is noted that the magnitude of the Bartlett window is almost unity for entire time index of M if N is very large

Performance of Blackman Tukey Method: Bias

- Then using eq(23 & 24), we will have:

$$E\ \hat{P}_{BT}(e^{j\omega}) = \frac{1}{2\pi} P_x(e^{j\omega}) * W_{BT}(e^{j\omega})$$

$$\approx \frac{1}{2\pi} P_x(e^{j\omega}) * W(e^{j\omega}) = \frac{1}{2\pi} \int_{-\pi}^{\pi} P_x(e^{ju})\ W(e^{j(\omega-u)})\ du \qquad (25)$$

Where:

$W(e^{j\omega})$ is the Fourier transform of the lag window $w(k)$

- So far we evaluated the expected value or the bias of the Blackman Turkey power spectrum estimate
- The variance analysis of the Blackman Tukey power spectrum estimate is more involved than its bias analysis

Performance of Blackman Tukey Method: Variance

- The variance of Blackman Tukey power spectrum estimate is given by:

$$Var \;\; \hat{P}_{BT}(e^{j\omega}) = E \;\; \hat{P}^2_{BT}(e^{j\omega}) - E^2 \;\; \hat{P}_{BT}(e^{j\omega}) \qquad (26)$$

- First, Let's find $E \;\; \hat{P}^2_{BT}(e^{j\omega})$, from eq (16) we have:

$$\hat{P}^2_{BT}(e^{j\omega}) = \left[\frac{1}{2\pi} \int_{-\pi}^{\pi} \hat{P}_{per}(e^{ju})\, W(e^{j(\omega-u)})\, du \right] \left[\frac{1}{2\pi} \int_{-\pi}^{\pi} \hat{P}_{per}(e^{jv})\, W(e^{j(\omega-v)})\, dv \right]$$

$$= \frac{1}{4\pi^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \hat{P}_{per}(e^{ju}) \hat{P}_{per}(e^{jv})\, W(e^{j(\omega-u)}) W(e^{j(\omega-v)})\, dudv \quad (27)$$

- Then the mean square value of Blackman Tukey estimate can be evaluated by taking the expected value of eq(27)

Performance of Blackman Tukey Method: Variance

- Then, we will have:

$$E\ \ \hat{P}_{BT}^{2}(e^{j\omega}) \ = \frac{1}{4\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} E\ \ \hat{P}_{per}(e^{ju})\hat{P}_{per}(e^{jv}) \ \ W(e^{j(\omega-u)})W(e^{j(\omega-v)})\, dudv \qquad (28)$$

- Using an approximation, we will have:

$$E\ \ \hat{P}_{per}(e^{ju})\hat{P}_{per}(e^{jv}) \ = P_x(e^{ju})P_x(e^{ju}) + P_x(e^{ju})P_x(e^{ju})\left[\frac{\sin N(u-v)/2}{N\sin(u-v)/2}\right]^2 \qquad (29)$$

- Then from eq(28 & 29), we will have

$$E\ \ \hat{P}_{BT}^{2}(e^{j\omega}) \ = \frac{1}{4\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} P_x(e^{ju})P_x(e^{ju})\ W(e^{j(\omega-u)})W(e^{j(\omega-v)})\, dudv$$

$$+\frac{1}{4\pi^2}\int_{-\pi}^{\pi} P_x(e^{ju})P_x(e^{ju})\left[\frac{\sin N(u-v)/2}{N\sin(u-v)/2}\right]^2 \ W(e^{j(\omega-u)})W(e^{j(\omega-v)})\, dudv \qquad (30)$$

Performance of Blackman Tukey Method: Variance

- Taking $u = v$, the first term of eq(30) can be simplified to:

$$\frac{1}{4\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} P_x(e^{ju})P_x(e^{ju})\, W(e^{j(\omega-u)})W(e^{j(\omega-v)})\, dudv$$

$$= \left[\frac{1}{2\pi}\int_{-\pi}^{\pi} P_x(e^{ju})\, W(e^{j(\omega-u)})\, du\right]\underbrace{\left[\frac{1}{2\pi}\int_{-\pi}^{\pi} P_x(e^{ju})\, W(e^{j(\omega-u)})\, du\right]}_{E\ \hat{P}_{BT}(e^{j\omega})}$$

$$= \left[\frac{1}{2\pi}\int_{-\pi}^{\pi} P_x(e^{ju})\, W(e^{j(\omega-u)})\, du\right]^2$$

$$= E^2\ \hat{P}_{BT}(e^{j\omega}) \qquad (31)$$

- From eq(30 and 31), we will have:

Performance of Blackman Tukey Method: Variance

$$E\left\{\hat{P}_{BT}^2(e^{j\omega})\right\} = E^2\left\{P_{BT}(e^{j\omega})\right\} + \frac{1}{4\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} P_x(e^{ju})P_x(e^{jv})\left[\frac{\sin N(u-v)/2}{N\sin(u-v)/2}\right]^2 W(e^{j(\omega-u)})W(e^{j(\omega-v)})\,du\,dv \quad (32)$$

- Then from eq (26) and eq(32), the variance becomes:

$$Var\left\{\hat{P}_{BT}(e^{j\omega})\right\} = E\left\{\hat{P}_{BT}^2(e^{j\omega})\right\} - E^2\left\{\hat{P}_{BT}(e^{j\omega})\right\}$$

$$= E^2\left\{P_{BT}(e^{j\omega})\right\} + \frac{1}{4\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} P_x(e^{ju})P_x(e^{jv})\left[\frac{\sin N(u-v)/2}{N\sin(u-v)/2}\right]^2 W(e^{j(\omega-u)})W(e^{j(\omega-v)})\,du\,dv - E^2\left\{P_{BT}(e^{j\omega})\right\}$$

$$= \frac{1}{4\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} P_x(e^{ju})P_x(e^{jv})\left[\frac{\sin N(u-v)/2}{N\sin(u-v)/2}\right]^2 W(e^{j(\omega-u)})W(e^{j(\omega-v)})\,du\,dv \qquad (33)$$

- Recalling the Bartlett window in frequency domain:

$$W_b(e^{j\omega}) = \frac{1}{N}\left[\frac{\sin\ N\omega/2}{\sin\ \omega/2}\right]^2 \qquad (34)$$

Performance of Blackman Tukey Method: Variance

- For large value of N, it is noted that the Bartlett window converge to an impulse with area $2\pi/N$ as::

$$W_b(e^{j\omega}) = \frac{1}{N}\left[\frac{\sin\ N\omega/2}{\sin\ \omega/2}\right]^2 \approx \frac{2\pi}{N}\delta(e^{j\omega}) \qquad (35)$$

- For large value of N the following term in eq (33) can be written in terms of Bartlett window evaluated at $\omega = u - v$ as:

$$\left[\frac{\sin N(u-v)/2}{N\sin(u-v)/2}\right]^2 = W_b(e^{j(u-v)}) = \frac{1}{N}\left[\frac{\sin\ N(u-v)/2}{\sin\ (u-v)/2}\right]^2 \approx \frac{2\pi}{N}\delta(e^{j(u-v)}) \qquad (36)$$

- Then from eq(33) and (36), the variance becomes:

Performance of Blackman Tukey Method: Variance

$$Var\ \hat{P}_{BT}(e^{j\omega}) == \frac{1}{4\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} P_x(e^{ju})P_x(e^{jv})\frac{2\pi}{N}\delta(e^{(u-v)})\, W(e^{j(\omega-u)})W(e^{j(\omega-v)})\, dudv \quad (37)$$

- Since:

$$\delta(e^{j(u-v)}) = \begin{cases} 1\ ; & \textbf{for } u = v \\ 0\ ; & Otherwise \end{cases} \quad (38)$$

- Hence, eq (37) can be simplified as:

$$Var\ \hat{P}_{BT}(e^{j\omega}) = \frac{1}{2\pi N}\int_{-\pi}^{\pi} P_x^2(e^{ju})\, W^2(e^{j(\omega-u)})\, du \quad (39)$$

- For large value of M, we may assume $P_x^2(e^{ju})$ is constant, therefore:

$$Var\ \hat{P}_{BT}(e^{j\omega}) \approx \frac{1}{2\pi N} P_x^2(e^{j\omega})\int_{-\pi}^{\pi} W^2(e^{j(\omega-u)})\, du \quad (40)$$

Performance of Blackman Tukey Method: Variance

- Applying Parzival's theorem, eq (40) can be written in alternative form as:

$$Var \ \hat{P}_{BT}(e^{j\omega}) \approx \frac{P_x^2(e^{j\omega})}{N} \sum_{k=-M}^{M} w^2(k) \qquad (41)$$

- The next figure shows the resolution of Blackman Turkey method while using hamming window for autocorrelation windowing for the process $x(n)$

$$x(n) = 15\sin(0.2\pi n + \phi_1) + 10\sin(0.22\pi n + \phi_2) + w(n)$$

For the following Parameters

$$N = 512$$

$$M = 400$$

$w(n)$ is unit variance white noise

Performance of Blackman Tukey Method: Variance

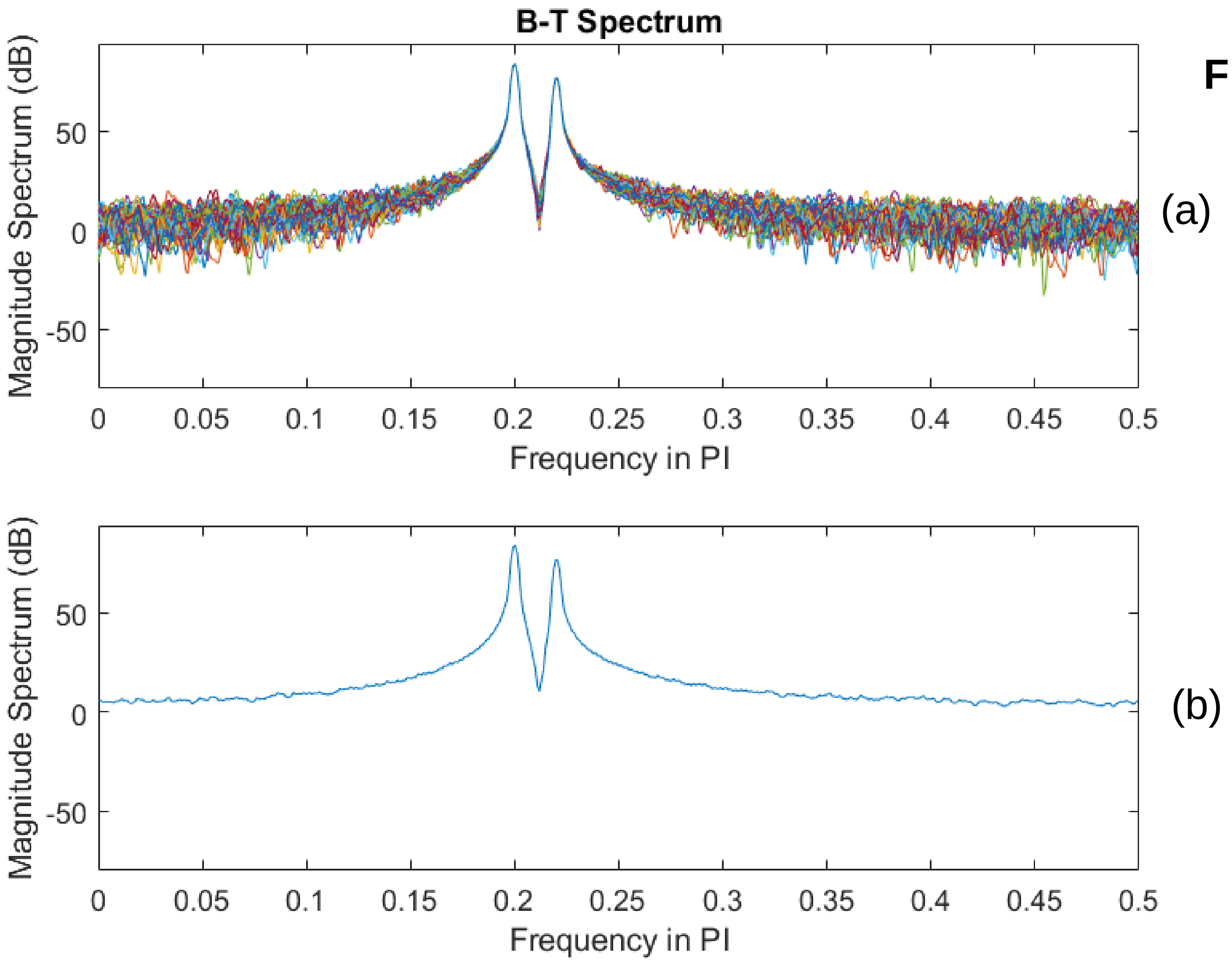

Figure 3: Blackman Tukey estimate of two sinusoids (a) Overlay plot of 50 Blackman Tukey estimate using N=512 and M=400 (b) the ensemble average

Source: MATLAB-generated plot using modified source code adapted from: S. Lakshmi, *"Spectrum-estimation,"* GitHub repository. https://github.com/srilakshmialla/Spectrum-estimation

Summary

- **Welch Method:**
  - ✓ Based on averaging of Modified Periodograms
  - ✓ The Modified Periodograms are computed from overlapping sequences
  - ✓ An asymptotically unbiased and consistent estimator of the power spectrum
- **Blackman Tukey Method**
  - ✓ Based on applying window function on the autocorrelation estimate to exclude unreliable estimation
  - ✓ The resolution of the estimator depends on the choice of window function

References

[1] Monson H. Hayes, “Statistical Digital Signal Processing and Modeling’’, John Wiley and sons, Pp.411, 1996.

[2] Charles W. Therrien,‘’ Discrete Random Signals and Statistical Signal Processing’’, Prentice Hall, Pp.593-595, 1992.

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Thank You!