Planar orbits and conserved quantities

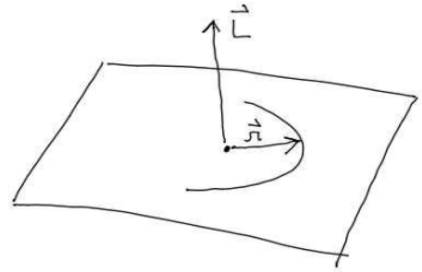
The solution $\vec{r}(t)$ depends on $f(r)$, but some aspects of $\vec{r}(t)$ turn out to be independent of $f(r)$, as we proceed to show.

Planar motion Since $f(r)$ is parallel to $\vec{r}$, it exerts no torque on the reduced mass μ.

Consequently the angular momentum does not change (since $\mathrm{d}\vec{L}/\mathrm{d}t = \vec{\tau} = 0$):

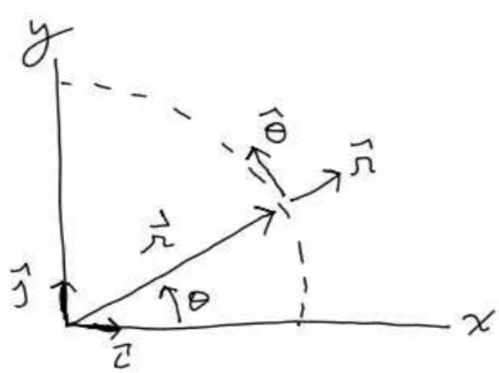
$$\vec{L} = \vec{r} \times \mu\vec{v} = \text{const.}, \qquad \vec{v} = \dot{\vec{r}}.$$

Since the cross product requires that $\vec{r} \perp \vec{L}$, constant $\vec{L}$ requires that $\vec{r}$ must always reside in a plane $\perp \vec{L}$ intersecting the origin.

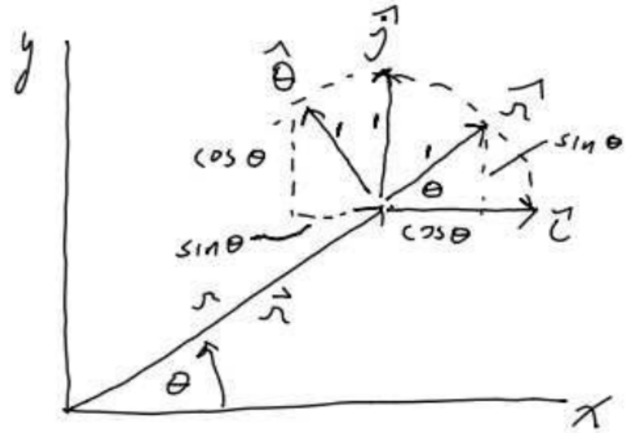


In other words, the motion is confined to a plane, and may therefore be described by just two coordinates.

Representation in polar coordinates We now choose coordinates such that this plane is the xy plane, and introduce polar coordinates r, θ. The associated unit vectors $\hat{r}, \hat{\theta}$ vary with position (unlike the usual Cartesian unit vectors $\hat{i}, \hat{j}$:



$\hat{r}$ and $\hat{\theta}$ are straightforwardly related to $\hat{i}$ and $\hat{j}$ graphically:



We thus have

$$\hat{r} = \hat{\imath}\cos\theta + \hat{\jmath}\sin\theta \tag{10}$$

$$\hat{\theta} = -\hat{\imath}\sin\theta + \hat{\jmath}\cos\theta. \tag{11}$$

We seek an expression for $\ddot{\vec{r}}$ in polar coordinates. Since $\hat{\imath}$ and $\hat{\jmath}$ are fixed unit vectors,

$$\frac{\mathrm{d}\hat{r}}{\mathrm{d}t} = -\hat{\imath}\dot{\theta}\sin\theta + \hat{\jmath}\dot{\theta}\cos\theta \tag{12}$$

$$= \dot{\theta}\hat{\theta} \tag{13}$$

and

$$\frac{\mathrm{d}\hat{\theta}}{\mathrm{d}t} = -\hat{\imath}\dot{\theta}\cos\theta - \hat{\jmath}\dot{\theta}\sin\theta \tag{14}$$

$$= -\dot{\theta}\hat{r}. \tag{15}$$

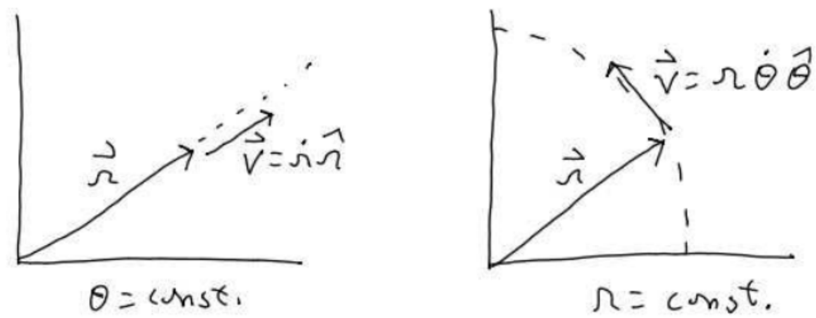
The velocity $\dot{\vec{r}}$ is then

$$\dot{\vec{r}} = \frac{\mathrm{d}}{\mathrm{d}t}(r\hat{r}) \tag{16}$$

$$= \dot{r}\hat{r} + r\frac{\mathrm{d}\hat{r}}{\mathrm{d}t} \tag{17}$$

$$= \dot{r}\hat{r} + r\dot{\theta}\hat{\theta}. \tag{18}$$

To see what this means, consider motion in which either θ or r is constant:



When $\theta = $ const., velocity is radial. Alternatively, when $r = $ const., velocity is tangential.

We proceed to use these relations to compute the acceleration:

$$\begin{aligned}\ddot{\vec{r}} &= \frac{\mathrm{d}}{\mathrm{d}t}(\dot{r}\hat{r} + r\dot{\theta}\hat{\theta}) \\ &= \ddot{r}\hat{r} + \dot{r}\frac{\mathrm{d}\hat{r}}{\mathrm{d}t} + \dot{r}\dot{\theta}\hat{\theta} + r\ddot{\theta}\hat{\theta} + r\dot{\theta}\frac{\mathrm{d}\hat{\theta}}{\mathrm{d}t}\end{aligned}$$

Inserting (13) and (15) we obtain

$$\ddot{\vec{r}} = \ddot{r}\hat{r} + \dot{r}\dot{\theta}\hat{\theta} + \dot{r}\dot{\theta}\hat{\theta} + r\ddot{\theta}\hat{\theta} - r\dot{\theta}^2\hat{r} \tag{19}$$

$$= (\ddot{r} - r\dot{\theta}^2)\hat{r} + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\hat{\theta} \tag{20}$$

The terms proportional to $\ddot{r}$ and $\ddot{\theta}$ represent acceleration in the radial and tangential directions, respectively. The term $-r\dot{\theta}^2\hat{r}$ is the *centripetal acceleration*, and the remaining term, $2\dot{r}\dot{\theta}\hat{\theta}$ is called the *Coriolis acceleration.*

We can now rewrite our one-body equation of motion (6) (i.e., $\mu\ddot{\vec{r}} = f(r)\hat{r}$) in polar coordinates.

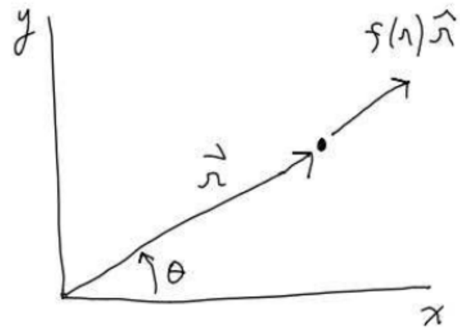
With respect to the radial coordinate $\hat{r}$, we have, after inserting (20), the radial equation of motion

$$\mu(\ddot{r} - r\dot{\theta}^2) = f(r). \tag{21}$$

Likewise, with respect to the angular coordinate $\hat{\theta}$ we have the tangential equation of motion

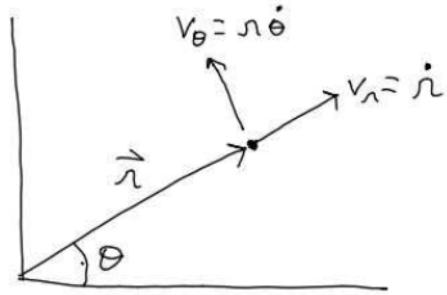
$$\mu(r\ddot{\theta} + 2\dot{r}\dot{\theta}) = 0. \tag{22}$$

These relations may look complicated, but they merely describe a particle of mass μ acted upon by a force in the radial direction:



Constants of motion: angular momentum and energy The foregoing development took advantage merely of the constant direction of the angular momentum $\vec{L}$. We now exploit its constant magnitude $l = |\vec{L}|$, and also use the conservation of the total energy E.

We decompose velocity $\vec{v}$ into radial and tangential components:



Since only the angular velocity v_θ contributes to l, we have, using the θ-component of $\dot{\vec{r}}$ from (18),

$$l = \mu r v_\theta = \mu r^2 \dot{\theta}. \tag{23}$$

(Note that computing time derivatives on the LHS and RHS above yields the tangential equation of motion (22).)

The total energy is the sum of the kinetic and potential energies. Using again equation (18), we have

$$\begin{aligned} E &= \frac{1}{2}\mu v^2 + U(r) \\ &= \frac{1}{2}\mu\left(\dot{r}^2 + r^2\dot{\theta}^2\right) + U(r). \end{aligned}$$

The potential energy $U(r)$ satisfies

$$U(r) - U(r_a) = -\int_{r_a}^{r} f(r)\mathrm{d}r$$

where $U(r_a)$ is a constant of no physical significance. [Note that, using (22), the radial equation of motion (21) is equivalent to $\mathrm{d}E/\mathrm{d}t = 0$.]

We substitute (23) for $\dot{\theta}$, thereby expressing energy in terms of the angular momentum l:

$$E = \frac{1}{2}\mu\dot{r}^2 + \frac{l^2}{2\mu r^2} + U(r).$$

We next define the *effective potential*

$$U_{\text{eff}}(r) = \frac{l^2}{2\mu r^2} + U(r)$$

wherein the first term on the RHS is called the *centrifugal potential*. Then

$$E = \frac{1}{2}\mu\dot{r}^2 + U_{\text{eff}}(r)$$

Rearranging, we have

$$\frac{\mathrm{d}r}{\mathrm{d}t} = \sqrt{\frac{2}{\mu}(E - U_{\text{eff}}(r))}.$$

We can also obtain $\mathrm{d}\theta/\mathrm{d}t$ directly from the angular momentum (23):

$$\frac{\mathrm{d}\theta}{\mathrm{d}t} = \frac{l}{\mu r^2}.$$

The orbit of the particle is given by r as a function of θ. We can obtain it by solving

$$\frac{\mathrm{d}\theta}{\mathrm{d}r} = \frac{\dot{\theta}}{\dot{r}} = \frac{l}{\mu r^2}\frac{1}{\sqrt{(2/\mu)(E - U_{\text{eff}}(r))}}. \tag{24}$$

This complete the formal solution of the two-body problem.

1.4.3 Elliptical orbits (Kepler's first law)

In considering Earth's orbit around the sun, note that the mass of the sun is about 330,000 times greater than that of the Earth.

Using the results of Section 1.4.1 and taking m_1 to be the mass of the Earth and m_2 the mass of the Sun, we conclude immediately that the center of mass $\vec{R}$ is essentially at the position of the Sun, which we take to be the origin.

Then, from equations (8) and (9),

$$\begin{aligned} \vec{r}_1 &= \left(\frac{m_2}{m_1 + m_2}\right)\vec{r} \\ &\simeq \vec{r} \end{aligned}$$

and

$$\begin{aligned} \vec{r}_2 &= -\left(\frac{m_1}{m_1 + m_2}\right)\vec{r} \\ &\simeq 0 \end{aligned}$$

since $m_1 \ll m_2$. From equation (7), we also have the reduced mass

$$\mu = \frac{m_1 m_2}{m_1 + m_2} = \frac{m_1}{m_1/m_2 + 1} \simeq m_1.$$

In other words, the Earth revolves around the sun as if the sun were fixed at the origin.

For planetary orbits, we have the gravitational interaction

$$U(r) = -G\frac{Mm}{r} \equiv \frac{-C}{r}, \tag{25}$$

where G is that gravitational constant, M the mass of the sun, and m the mass of the planet.

This potential ignores the interactions with other planets. That is, Earth's orbit is not purely the result of a two-body interaction. Indeed, perturbations due to interactions with other bodies are the principal cause of the Milankovitch oscillations—but to understand how these perturbations work, we must first understand the unperturbed problem.

The effective potential is now

$$U_{\text{eff}}(r) = \frac{l^2}{2\mu r^2} - \frac{C}{r},$$

where we retain the use of μ. Inserting into (24) and integrating, we have

$$\begin{aligned}\theta - \theta_0 &= l \int \frac{\mathrm{d}r}{r\sqrt{2\mu E r^2 + 2\mu C r - l^2}} \\ &= \sin^{-1}\left(\frac{\mu C r - l^2}{r\sqrt{\mu^2 C^2 + 2\mu E l^2}}\right),\end{aligned}$$

as may be found, e.g., in a table of integrals. We rewrite the latter expression as

$$\mu C r - l^2 = r\sqrt{\mu^2 C^2 + 2\mu E l^2}\,\sin(\theta - \theta_0).$$

We then solve for r:

$$r = \frac{l^2/\mu C}{1 - \sqrt{1 + 2El^2/\mu C^2}\,\sin(\theta - \theta_0)}.$$

We take $\theta_0 = -\pi/2$ so that $\sin(\theta - \theta_0) = \cos\theta$.

We also define the parameters

$$r_0 = \frac{l^2}{\mu C} \tag{26}$$

and

$$\varepsilon = \sqrt{1 + \frac{2El^2}{\mu C^2}}. \tag{27}$$

When $\varepsilon = 0$, r_0 is the radius of the circular orbit corresponding to l, μ, and C.

The parameter ε is called the *eccentricity* of the orbit. To see why, we rewrite r in terms of r_0 and ε:

$$r = \frac{r_0}{1 - \varepsilon\cos\theta}. \tag{28}$$

We next revert to cartesian coordinates $x = r\cos\theta$, $y = r\sin\theta$. From above, we have that

$$r - \varepsilon r\cos\theta = r_0$$

which is expressed in cartesian coordinates as

$$\sqrt{x^2 + y^2} = r_0 + \varepsilon x.$$

Squaring both sides,

$$x^2 + y^2 = r_0^2 + 2r_0\varepsilon x + \varepsilon^2 x^2$$

and therefore

$$(1 - \varepsilon^2)x^2 - 2r_0\varepsilon x + y^2 = r_0^2.$$

The shape of the orbit depends on ε:

- $\varepsilon > 1$ corresponds to a *hyperbola*. Equation (27) then requires $E > 0$.
- $\varepsilon = 1$ corresponds to a parabola (and $E = 0$).
- $0 \leq \varepsilon < 1$ corresponds to an *ellipse*, with

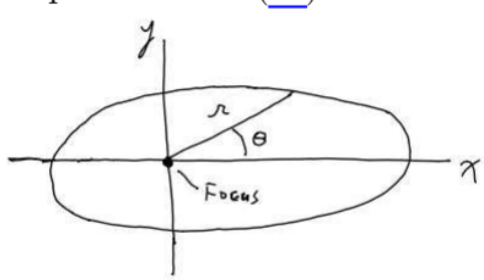
  $$-\frac{\mu C^2}{2l^2} \leq E < 0.$$

  The origin is one focus of the ellipse. When $\varepsilon = 0$ the ellipse becomes a *circle*.

The case $0 \leq \varepsilon < 1$ corresponds to *Kepler's first law*: planetary orbits are ellipses with the sun at one of the two foci.

The properties of elliptical orbits are of much interest to (Earth's) orbital oscillations.

We return to the polar representation (28).

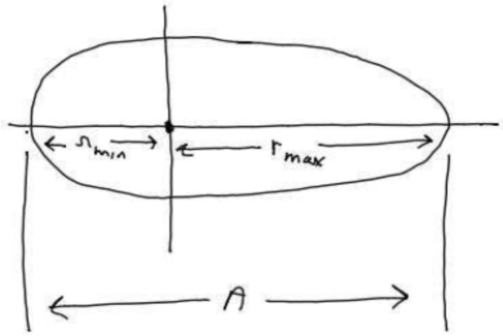


We see that the maximum value of r occurs at $\theta = 0$:

$$r_{\max} = \frac{r_0}{1 - \varepsilon}$$

The minimum value of r occurs at $\theta = \pi$:

$$r_{\min} = \frac{r_0}{1+\varepsilon}$$



The length A of the major axis is therefore

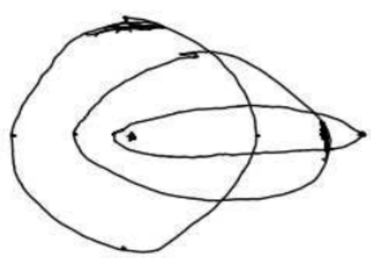
$$\begin{aligned} A &= r_{\min} + r_{\max} \\ &= r_0 \left(\frac{1}{1+\varepsilon} + \frac{1}{1-\varepsilon} \right) \\ &= \frac{2r_0}{1-\varepsilon^2}. \end{aligned}$$

Substituting equations (26) and (27) above, we obtain

$$A = \frac{2l^2/(\mu C)}{1 - [1 + 2El^2/(\mu C^2)]} \tag{29}$$

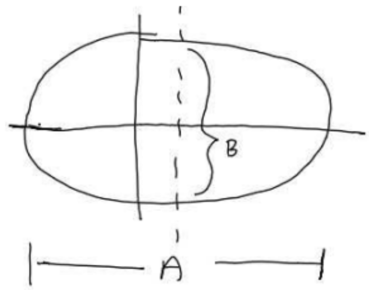
$$= -\frac{C}{E}. \tag{30}$$

Thus the length of the major axis is independent of the angular momentum ℓ and orbits with the same major axis have the same energy E, e.g.:



The minor axis of the ellipse is easily shown to be

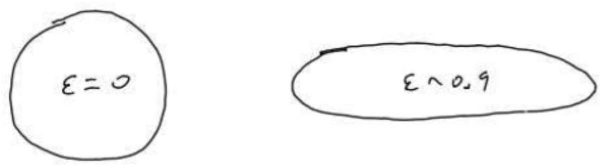
$$B = \frac{2r_0}{\sqrt{1-\varepsilon^2}}. \tag{31}$$



The ratio of the lengths of the major and minor axes is

$$\frac{A}{B} = \frac{2r_0/(1-\varepsilon^2)}{2r_0/\sqrt{1-\varepsilon^2}} = \frac{1}{\sqrt{1-\varepsilon^2}} \tag{32}$$

As ε increases towards 1, the ellipse becomes more elongate:



The present eccentricity of Earth's orbit is small: $\varepsilon = 0.016722$. Thus

$$\left.\frac{A}{B}\right|_{\mathbf{Earth}} = 1.00014,$$

showing that the Earth's orbit is circular within 0.014%..

The difference between the maximum and minimum distances from the sun, however, varies more. Relative to the length of the semi-major axis, we have

$$\frac{r_{\max} - r_{\min}}{A/2} = \frac{2\varepsilon r_0/(1-\varepsilon^2)}{r_0/(1-\varepsilon^2)} = 2\varepsilon,$$

which is about 3.3% for Earth's orbit.

This small difference accounts for changes in solar insolation, as we discuss in Section 1.5.

But first we discuss how the eccentricity or Earth's orbit can change.

1.4.4 Relation of eccentricity to angular momentum

We rewrite the eccentricity equation (27) as

$$\varepsilon^2 = 1 + \frac{2El^2}{m^3M^2G^2}.$$

For elliptical orbits, the energy E is negative.

A classical result in celestial mechanics shows that, when a planet's orbit is perturbed by another body, the major axis A remains invariant to first order in the masses, except for short-period oscillations that do not affect mean behavior.[‡]

Therefore, via equation (30), E can be taken to be effectively constant. Thus

$$k \equiv \frac{-2E}{m^3M^2G^2} \simeq \text{const} > 0,$$

and by rewriting eccentricity as

$$\varepsilon^2 = 1 - kl^2$$

we find that the only way to change ε is to change the magnitude of the angular momentum, l.

Consider the extreme cases:

- $\varepsilon \to 1$. Then $l \to 0$, because the object is falling nearly directly towards the sun, with no transverse velocity.
- $\varepsilon = 0$. Then $l = l_{\max} = k^{-1/2}$ and the orbit is circular.

Thus any force that removes angular momentum makes the orbit more eccentric, and any force that adds it makes the orbit more circular.

[‡]Specifically, A exhibits no *secular* variations that grow like t or $t \sin t$ to first-order in the masses, a result due to Lagrange, following earlier results of Laplace. Poisson later showed that no purely secular variations (growing without oscillating) occur at second order. Periodic oscillations of A do occur at first order, but in the solar system these are all at much shorter periods than concern us here.

The angular momentum $\vec{L}$ changes due to an applied torque $\vec{\tau}$; i.e.,

$$\frac{\mathrm{d}\vec{L}}{\mathrm{d}t} = \vec{\tau}.$$

Torque on Earth's orbit is produced by planets pulling on the Earth and Sun asymmetrically.

The major torques are those of Jupiter, because it is so large, and Venus, because it is so close.

As a consequence, the eccentricity of Earth's orbit varies between about 0 and 0.05, with periods of 95, 125, and 400 Kyr.

1.5 Insolation

1.5.1 Daily and yearly insolation

The average flux of solar energy at the top of the Earth's atmosphere is

$$S = 1360 \text{ Watts/m}^2.$$

This quantity, called the *solar constant*, is the solar electromagnetic radiation per unit area if it were arriving at normal incidence.

Taking the Earth's radius to be R_e, we define

$$W = \text{total solar energy flux received by Earth} = \pi R_e^2 S.$$

But this flux is spread out over an area of size $4\pi R_e^2$.

Dividing the total flux by the area of the earth, we obtain the average daily *insolation*

$$I = \frac{W}{4\pi R_e^2} = \frac{S}{4} = 340 \text{ W/m}^2.$$

The actual insolation on any given day depends on the distance from the Sun. Let

$$S_a = \text{energy flux received a distance } a \text{ from the sun,}$$

where $a = A/2$, the semi-major axis of the elliptical orbit.

When the earth is a distance r from the sun, the average daily insolation is then

$$I(r) = \frac{S_a}{4}\left(\frac{a}{r}\right)^2.$$

where the quadratic factor arises from the spherical spreading of the Sun's radiation.

Over a year of length T, the average insolation is

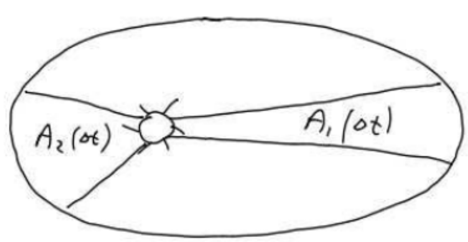
$$I_T = \frac{1}{T}\int_0^T I[r(t)]\mathrm{d}t = \frac{S_a}{4T}\int_0^T \left(\frac{a}{r}\right)^2 \mathrm{d}t. \tag{33}$$

To calculate this integral, we must first derive *Kepler's second law.*

1.5.2 Kepler's second law

(We have already derived Kepler's first law in Section 1.4.3: planetary orbits are elliptical, a conseqence of the $1/r^2$ gravitational force.)

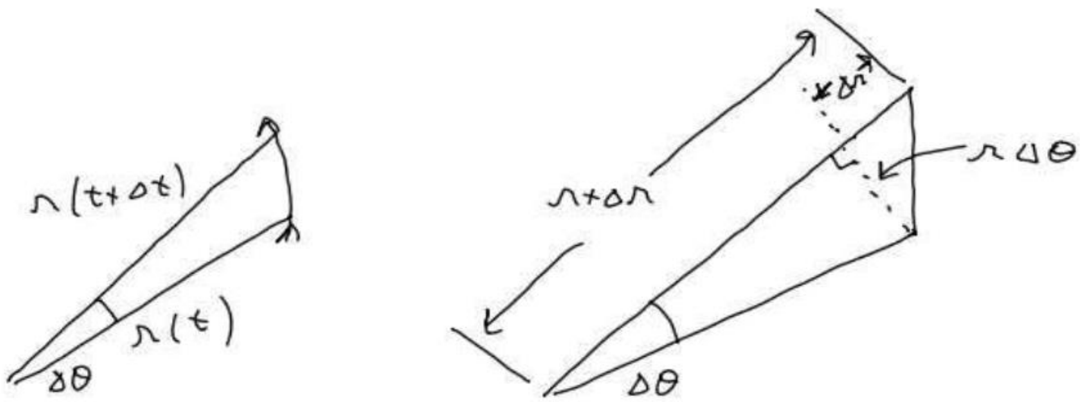
Kepler's second law states that the area $\mathcal{A}$ swept out by the radius vector from the sun to a planet in a given length of time is constant throughout the orbit:



In other words, $\mathcal{A}_1 = \mathcal{A}_2$ and, more generally, $\dfrac{\mathrm{d}\mathcal{A}}{\mathrm{d}t} = \text{const.}$.

To show this, we note that a small change in area, $\Delta\mathcal{A}$, due to small change Δr and $\Delta\theta$ is

$$\begin{aligned}\Delta\mathcal{A} &\simeq \frac{1}{2}(r+\Delta r)(r\Delta\theta)\\ &= \frac{1}{2}r^2\Delta\theta + \frac{1}{2}r\Delta r\Delta\theta\end{aligned}$$



Then

$$\begin{aligned}\frac{\mathrm{d}\mathcal{A}}{\mathrm{d}t} &= \lim_{\Delta t\to 0}\frac{\Delta\mathcal{A}}{\Delta t}\\ &= \lim_{\Delta t\to 0}\frac{1}{2}\left(r^2\frac{\Delta\theta}{\Delta t} + r\frac{\Delta r\Delta\theta}{\Delta t}\right)\\ &= \frac{1}{2}r^2\frac{\mathrm{d}\theta}{\mathrm{d}t}\end{aligned}$$

where we have neglected the small second order term representing the tiny triangle.

Now note that the angular momentum of the Earth relative to the sun is

$$\vec{L} = \vec{r}\times m\vec{v}$$

From equation (18), the velocity

$$\vec{v} = \dot{r}\hat{r} + r\dot{\theta}\hat{\theta}.$$

Consequently

$$\begin{aligned}\vec{L} &= \vec{r}\times m(\dot{r}\hat{r} + r\dot{\theta}\hat{\theta})\\ &= mr^2\frac{\mathrm{d}\theta}{\mathrm{d}t}\hat{k}\end{aligned}$$

since $\hat{r} \times \hat{\theta} = \hat{k}$. Substituting the expression above into that for $\mathrm{d}\mathcal{A}/\mathrm{d}t$, we have

$$\frac{\mathrm{d}\mathcal{A}}{\mathrm{d}t} = \frac{l}{2m} = \text{const.} \tag{34}$$

Recalling that the angular momentum l is a constant for the orbit (Section 1.4.2), we thus arrive at Kepler's second law.

1.5.3 Relation of insolation to eccentricity

We return now to the computation of the annually averaged insolation I_T, and thus the integral $\int (a/r)^2 \mathrm{d}t$ of (33). From the results we have just obtained, we have

$$\begin{aligned} \frac{r^2}{2}\frac{\mathrm{d}\theta}{\mathrm{d}t} &= \frac{\text{area of ellipse}}{T} \\ &= \frac{\pi ab}{T}, \end{aligned}$$

where $b = B/2$, the semi-minor axis, and T is the duration of a year.

From equation (32), we have $b = a\sqrt{1-\varepsilon^2}$; therefore

$$\frac{r^2}{2}\frac{\mathrm{d}\theta}{\mathrm{d}t} = \frac{\pi a^2\sqrt{1-\varepsilon^2}}{T}.$$

We rewrite this expression as

$$\left(\frac{a^2}{r^2}\right)\mathrm{d}t = \frac{T}{2\pi\sqrt{1-\varepsilon^2}}\,\mathrm{d}\theta$$

Substituting this result into equation (33), the annually averaged insolation, we obtain

$$I_T = \frac{S_a}{4T}\int_0^{2\pi} \frac{T}{2\pi\sqrt{1-\varepsilon^2}}\,\mathrm{d}\theta$$

Since for Earth's orbit, ε varies only from about 0.0 to 0.05 in 100 Kyr, to good approximation it is constant over one year (T). Thus

$$I_T = \frac{S_a}{4\sqrt{1-\varepsilon^2}}. \tag{35}$$

We previously observed, in Section 1.4.4, that the major axis A is effectively constant. Consequently S_a can be taken constant.

The annually averaged insolation I_T therefore depends only on the eccentricity.

Since ε is small, we can expand I_T to second order about $\varepsilon = 0$:

$$\begin{aligned} I_T(\varepsilon) &= \frac{S_a}{4}\left(1 + \frac{\mathrm{d}}{\mathrm{d}\varepsilon}\frac{1}{\sqrt{1-\varepsilon^2}}\bigg|_{\varepsilon=0} \cdot \varepsilon + \frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}\varepsilon^2}\frac{1}{\sqrt{1-\varepsilon^2}}\bigg|_{\varepsilon=0} \cdot \varepsilon^2 + \dots\right) \\ &= \frac{S_a}{4}\left(1 + \frac{\varepsilon^2}{2} + \mathcal{O}(\varepsilon^4)\right). \end{aligned}$$

Thus increasing eccentricity from 0 to 0.05 produces an increase in the relative yearly insolation by a factor of about $0.05^2/2$, or about 0.1%.

This small change can be understood from the figure below equation (30): as eccentricity increases, about half the orbit becomes further away from the Sun, while the other half is closer. Thus the changes almost cancel.

We can get a sense of what the actual changes mean by recalling, from the beginning of this section, that the average daily insolation is 340 $\mathrm{W/m^2}$.

Thus the increase in daily insolation due to increasing eccentricity is much less than 1 $\mathrm{W/m^2}$.

In contrast, the effective change in radiative forcing due to other changes is much larger:

effect	equivalent radiative force ($\mathrm{W/m^2}$)
average daily insolation	340
average reflected insolation (albedo)	−53.5
clouds	−28
doubling CO_2	4

Consequently changing eccentricity has only a minor impact on radiative forcing.

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