Statistical Digital Signal Processing

Week 12 Parametric Spectrum Estimation

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Previous Topic (Week-11)

Welch and Blackman Tukey methods

- Welch Method
- Performance of Welch Method (Bias and Variance)
- Blackman Tukey method
- Performance of Blackman Tukey Method (Bias and Variance)

Lecture Learning Outcomes

1. Explain the fundamental concepts and applications of parametric spectrum estimation in signal processing
2. Describe the major steps involved in parametric spectrum estimation methods
3. Analyze and apply the Autoregressive (AR) model for spectrum estimation problems
4. Evaluate the principles and development of Moving Average (MA) spectrum estimation technique
5. Compare and apply Autoregressive Moving Average (ARMA) spectrum estimation methods for modeling and analyzing signals

Week 12: Parametric Spectrum Estimation

Outline

- Parametric Spectrum Estimation: Introduction
- Parametric Spectrum Estimation Steps
- Autoregressive (AR) Spectrum Estimation
- Moving Average (MA) Spectrum Estimation
- Autoregressive Moving Average (ARMA) Spectrum Estimation

Parametric Spectrum Estimation: Introduction

- So far we have seen different non-parametric spectrum estimation methods
- Nonparametric spectrum estimation methods cannot incorporate prior knowledge about the signal generation process into the estimation procedure
- A key limitation of nonparametric spectrum estimation methods is their inability to utilize available information about the underlying process during estimation
- In some applications this may be an important limitation, particularly when some knowledge is available about how the data samples are generated
- In applications such as speech processing, the vocal tract can be modeled as an acoustic tube, leading to an autoregressive (AR) representation of the speech signal
- Consequently, for short intervals where the speech signal is approximately stationary, the spectrum can be modeled using an autoregressive form as follows:

Parametric Spectrum Estimation: Introduction

$$P_x(e^{j\omega}) = \frac{|b(0)|^2}{\left|1+\sum_{k=1}^{p} a_p(k)e^{-jk\omega}\right|^2} \qquad (1)$$

- However, with non-parametric approach such as periodogram, the estimated power spectrum has the form that is consistent with the moving average process:

$$\hat{P}_{per}(e^{j\omega}) = |X_N(e^{j\omega})|^2 = \sum_{k=-N}^{N} \hat{r}_x(k)e^{-jk\omega} \qquad (2)$$

- Incorporating the process model into spectrum estimation can
  - ❖ Improve accuracy
  - ❖ Provide higher-resolution

Parametric Spectrum Estimation Steps

- Incorporating the process model into spectrum estimation may be easily done using a parametric approach to spectrum estimation
- **Parametric Approach:** Steps of developing parametric method are as follows:

Step-1: Model Selection

- Choose a suitable model for the Process
- Selection may based on:
  - Prior knowledge of how the process is generated
  - Experimental results indicating that a particular model "works well."

- Parametric models that are commonly used include [1]:
  - Autoregressive (AR)
  - moving average (MA)
  - Autoregressive moving average (ARMA)

Parametric Spectrum Estimation Steps

- Once the model is selected the next step will be:

Step-2: Estimate Model Parameters

- Use the observed data to estimate model coefficients
  - Coefficients such as $\hat{a}_p(k)$
  - and $\hat{b}_q(k)$

- Once the model parameters are estimated, the final step will be :

Step-3: Estimate the Power Spectrum

- Substitute estimated parameters into the parametric spectrum equation
- Obtain the final spectrum estimate

Parametric Spectrum Estimation

- For an autoregressive moving average model, with estimates $\hat{a}_p(k)$ and $\hat{b}_q(k)$ of the model parameters, the spectrum estimate is given by:

$$\hat{P}_x(e^{j\omega}) = \frac{\left|\sum_{k=0}^{q} \hat{b}_q(k) e^{-jk\omega}\right|^2}{\left|1 + \sum_{k=1}^{p} \hat{a}_p(k) e^{-jk\omega}\right|^2} \qquad (3)$$

- Parametric methods can greatly improve spectrum resolution, but using an inappropriate model may lead to inaccurate or misleading estimates
- For example, let's consider the power spectrum estimate of the process $x(n)$ comprising two sinusoids and unit variance white noise $w(n)$

Parametric Spectrum Estimation

$$x(n) = 5\sin(0.45\pi n + \phi_1) + 5\sin(0.55\pi n + \phi_2) + w(n) \qquad (4)$$

- Figure 1 illustrates the comparison of AR model and Blackman Tukey method

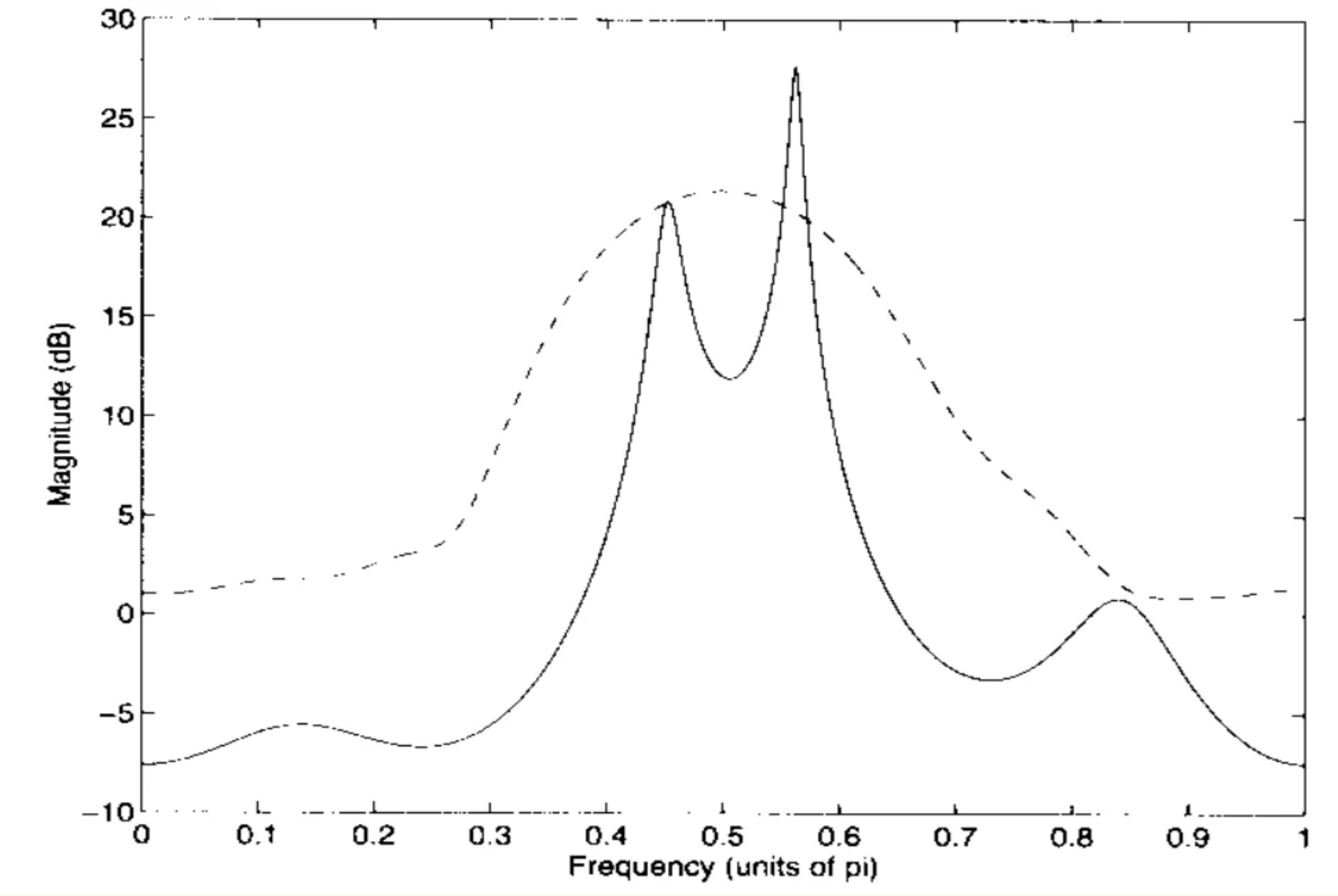


Figure 1: Spectrum estimation for two sennosides process using AR spectrum estimation (solid line) and Blackman Tukey method (dashed line) for N=64

Source: "Parametric Spectrum estimation (AR model) Vs Non-Parametric Spectrum estimation (Blackman Tukey)", dedicated-lark. https://custom-images.strikinglycdn.com/res/hrscywv4p/image/upload/c_limit,fl_lossy,h_9000,w_1200,f_auto,q_auto/24944559/970284_761686.png

Parametric Spectrum Estimation

- As revealed by Figure 1, the Parametric approach using all pole /AR model provides much better resolution compared to the Blackman Tukey method
- To see the effect of model choice in parametric approach, lets consider estimating the power spectrum of 2nd order moving average Process, $x(n)$, given by:

$$x(n) = w(n) - w(n-2) \qquad (5)$$

Where:

$w(n)$ is a unit variance white noise

- The true power spectrum of $x(n)$ is:

$$P_x(e^{j\omega}) = 2 - 2\cos(2\omega) \qquad (6)$$

Parametric Spectrum Estimation

- Figure 2 illustrates the comparison of AR model and Blackman Tukey method for estimating the power spectrum of the MA(2) process $x(n)$

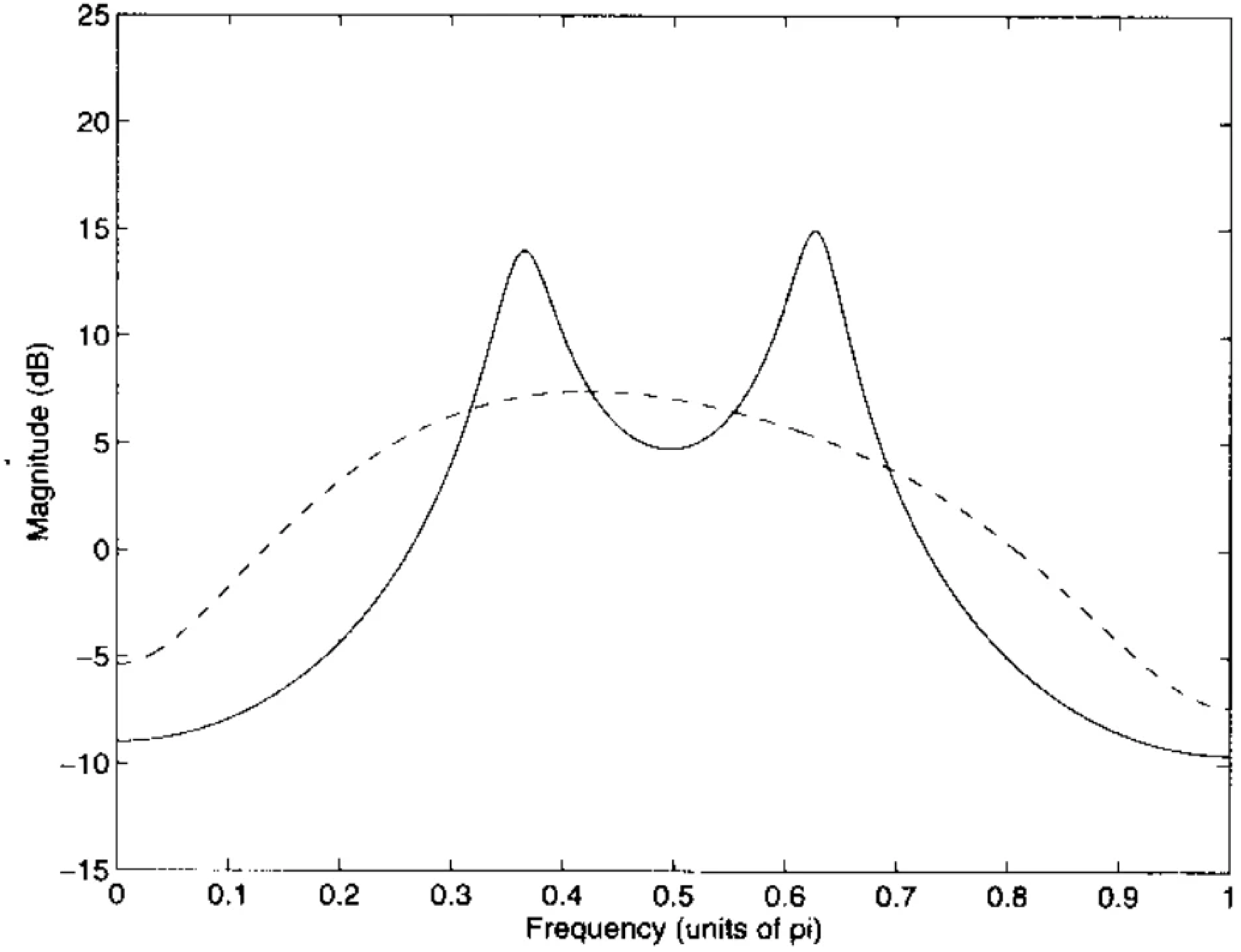


Figure 2: Spectrum estimation for MA(2) process using AR spectrum estimation (solid line) and Blackman Tukey method (dashed line) for N=64

Source: "Parametric Spectrum estimation (AR model) Vs Non-Parametric Spectrum estimation (Blackman Tukey) , for MA(2) Process", dedicated-lark. https://custom-images.strikinglycdn.com/res/hrscywv4p/image/upload/c_limit,fl_lossy,h_9000,w_1200,f_auto,q_auto/24944559/131153_100124.png

Parametric Spectrum Estimation

- As illustrated on Figure 2 illustrates, the parametric model inaccurately represent the true power spectrum since AR model is used to estimate the power spectrum of MA process.
- Therefore, choosing appropriate model which can work for the process is important to accurately estimate its power spectrum
- Unlike parametric methods, the Blackman–Tukey method makes no assumptions about the process and yields a more accurate power spectrum estimate
- In today's lecture we will cover spectrum estimation using:
    - Autoregressive (AR) Model
    - moving average (MA) Model
    - Autoregressive moving average (ARMA) Model

Autoregressive Spectrum Estimation

- An autoregressive process, $x(n)$, can be represented by the output of all-pole filter with an input of unit variance white noise
- The power spectrum of the $p^{th} - order$ autoregressive process is given by:

$$P_x(e^{j\omega}) = \frac{|b(0)|^2}{\left|1 + \sum_{k=1}^{p} a_p(k) e^{-jk\omega}\right|^2} \qquad (6)$$

- Therefore, if $b(0)$ and $a_p(k)$ can be estimated from the available data, the estimated power spectrum can be given as:

$$\hat{P}_{AR}(e^{j\omega}) = \frac{|\hat{b}(0)|^2}{\left|1 + \sum_{k=1}^{p} \hat{a}_p(k) e^{-jk\omega}\right|^2} \qquad (7)$$

Autoregressive Spectrum Estimation

- From eq(7), it is noted that the accuracy of $\hat{P}_{AR}(e^{j\omega})$ depends on the accuracy of model parameters estimation
- In addition, the accuracy of $\hat{P}_{AR}(e^{j\omega})$ is also depend on whether or not an autoregressive model is consistent with the way in which the data is generated
- For example if eq(7) is applied for moving average process, we will obtain poor estimation for the power spectrum
- Autoregressive (AR) spectrum estimation requires determining an all-pole model of the process using various parameter estimation techniques
- Once the all-pole parameters are estimated, all methods compute the power spectrum in the same manner using eq(7)
- Next, we will focus on different AR modeling techniques and their properties in spectrum estimation

AR Spectrum Estimation: Autocorrelation Method

- In the **autocorrelation method** of all pole modeling, the estimated AR coefficients $\hat{a}_p(k)$ can found by solving the following autocorrelation normal equations

$$\begin{bmatrix} \hat{r}_x(0) & \hat{r}_x^*(1) & \hat{r}_x^*(2) \cdots & \hat{r}_x^*(p) \\ \hat{r}_x(1) & \hat{r}_x(0) & \hat{r}_x^*(1) \cdots & \hat{r}_x^*(p-1) \\ \hat{r}_x(2) & \hat{r}_x(1) & \hat{r}_x(0) \cdots & \hat{r}_x^*(p-2) \\ \vdots & \vdots & \vdots \quad \cdots & \vdots \\ \hat{r}_x(p) & \hat{r}_x(p-1) & \hat{r}_x(p-2) \cdots & \hat{r}_x(0) \end{bmatrix} \begin{bmatrix} 1 \\ \hat{a}_p(1) \\ \hat{a}_p(2) \\ \vdots \\ \hat{a}_p(p) \end{bmatrix} = \varepsilon_p \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \qquad (8)$$

- Where $\hat{r}_x(k)$ in eq(8) is the estimated autocorrelation from the data and given by:

$$\hat{r}_x(k) = \frac{1}{N} \sum_{n=0}^{N-1-k} x(n+k)x^*(n) \;\; ; \;\; k = 0,1,\ldots,p \qquad (9)$$

AR Spectrum Estimation: Autocorrelation Method

- Then solving for $\hat{a}_p(k)$ using eq (8) and setting:

$$\left|\hat{b}_0(0)\right|^2 = \varepsilon_p = \hat{r}_x(0) + \sum_{k=1}^{p} \hat{a}_p(k)\, \hat{r}_x^*(k) \qquad (10)$$

- Now since both $\hat{a}_p(k)$ and $\left|\hat{b}_0(0)\right|^2$ are known, we can put the them in eq(7) to obtain the estimate of the power spectrum, and this method some times is called **Yule-Walker Method** [2]
- The autocorrelation matrix $\hat{\mathbf{R}}_x$ in eq (8) is Toeplitz, hence, the Levinson-Durbin recursion can be used to solve these equations to find $\hat{a}_p(k)$
- While estimating the autocorrelation using eq(8), the autocorrelation method applies rectangular window to the data
- As a result, the autocorrelation method provide lower resolution compared to approaches which do not use windowing such as **Covariance method**

AR Spectrum Estimation: Autocorrelation Method

- The other limitation of the autocorrelation method is the occurrence of **spectral line splitting**
- **Spectral line splitting:** splitting of single spectral peak in to two separate peaks
- Spectral line splitting occurs when the process $x(n)$ is **over-modeled** or when the order of the model, p, is too large.
- To visualize the spectral line splitting, let's see an AR(2) process that is generated according to:

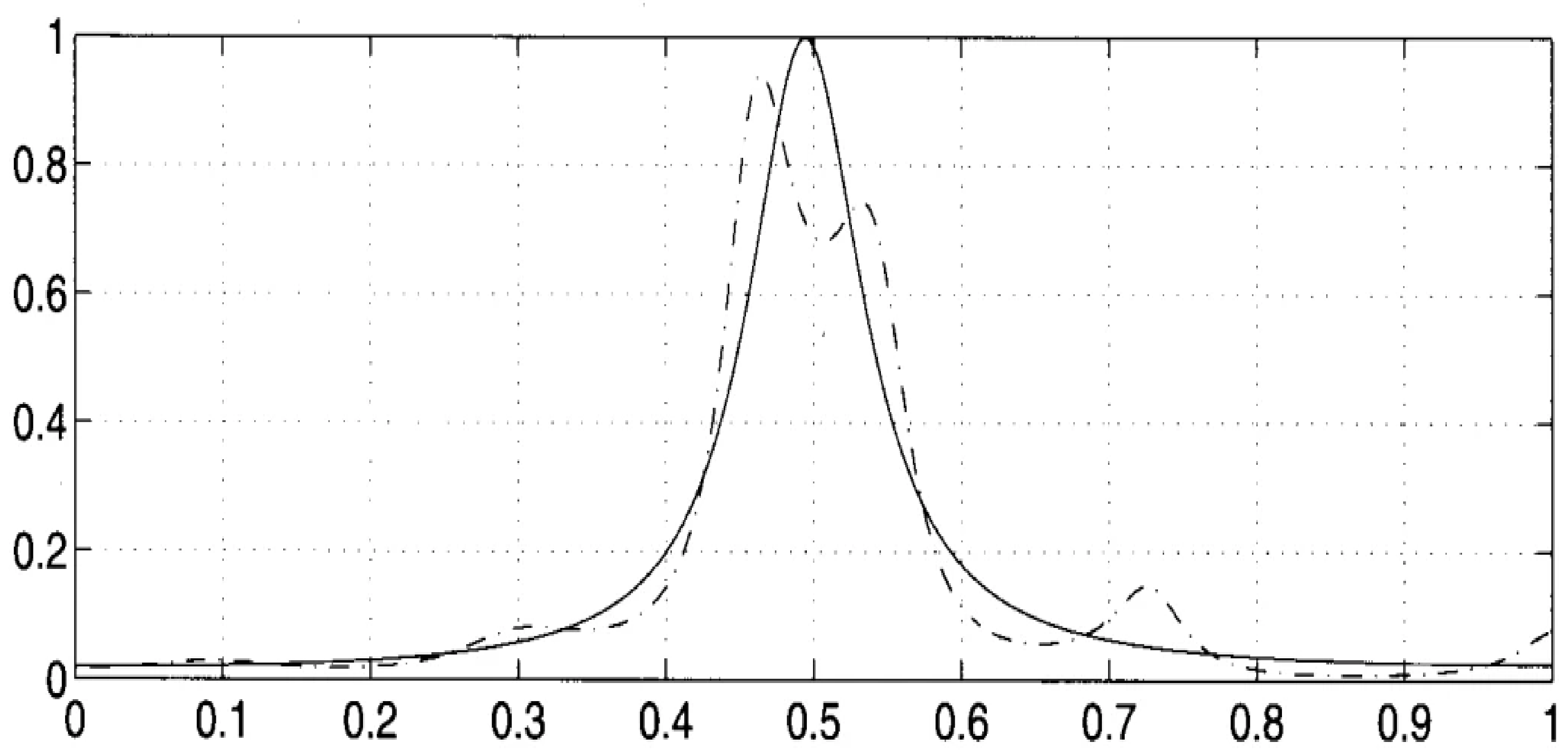
$$x(n) = -0.9x(n-2) + w(n) \qquad (11)$$

Where:

$w(n)$ is a unit variance white noise

- The spectrum estimate for $p = 4$ and $p = 12$ with $N = 64$ data values is given in the next figure

AR Spectrum Estimation: Autocorrelation Method



Frequency in PI

Figure 3: Spectral line splitting of AR(2) process using AR spectrum estimation, p=4 (solid line), p=12 (dashed line)

Source: "Spectral Line Splitting, AR(2) Process", dedicated-lark. https://custom-images.strikinglycdn.com/res/hrscywv4p/image/upload/c_limit,fl_lossy,h_9000,w_1200,f_auto,q_auto/24944559/704226_510121.png

- The true power spectrum has single peak at $\omega = \pi/2$

AR Spectrum Estimation: Covariance Method

- An alternative technique for estimating AR parameters by solving a set of linear equation is **Covariance Method**
- The covariance method requires solving the following set of linear equations

$$\begin{bmatrix} \hat{r}_x(1,1) & \hat{r}_x(2,1) & \cdots & \hat{r}_x(p,1) \\ \hat{r}_x(1,2) & \hat{r}_x(2,2) & \cdots & \hat{r}_x(p,2) \\ \hat{r}_x(1,3) & \hat{r}_x(2,3) & \cdots & \hat{r}_x(p,3) \\ \vdots & \vdots & \cdots & \vdots \\ \hat{r}_x(1,p) & \hat{r}_x(2,p) & \cdots & \hat{r}_x(p,p) \end{bmatrix} \begin{bmatrix} \hat{a}_p(1) \\ \hat{a}_p(2) \\ \hat{a}_p(3) \\ \vdots \\ \hat{a}_p(p) \end{bmatrix} = - \begin{bmatrix} \hat{r}_x(0,1) \\ \hat{r}_x(0,2) \\ \hat{r}_x(0,3) \\ \vdots \\ \hat{r}_x(0,p) \end{bmatrix} \qquad (12)$$

Where:

$$\hat{r}_x(k,l) = \sum_{n=p}^{N-1} x(n-l)x^*(n-k) \qquad (13)$$

AR Spectrum Estimation: Covariance Method

- Unlike the autocorrelation method, eq(12) is not Toeplitz since $\hat{r}_x(k,l) \neq \hat{r}_x(k+1, l+1)$
- The advantage of covariance method over the autocorrelation method is, the covariance method does not use data windowing in the formation of the autocorrelation estimate $\hat{r}_x(k,l)$
- Therefore, for short data records the covariance method generally produces higher resolution spectrum estimates than the autocorrelation method
- However, as the data record length increases and becomes large compared to the model order, **N≫p**, the effect of the data window becomes small and the difference between the two approaches becomes negligible.
- To see the performance comparison of Autocorrelation and Covariance methods, let's consider the power spectrum estimate of AR(4) process:

$$x(n) = 2.7377x(n-1) - 3.7476x(n-2) + 2.6293x(n-3) - 0.9224x(n-4) + w(n) \qquad (14)$$

AR Spectrum Estimation: Covariance Method

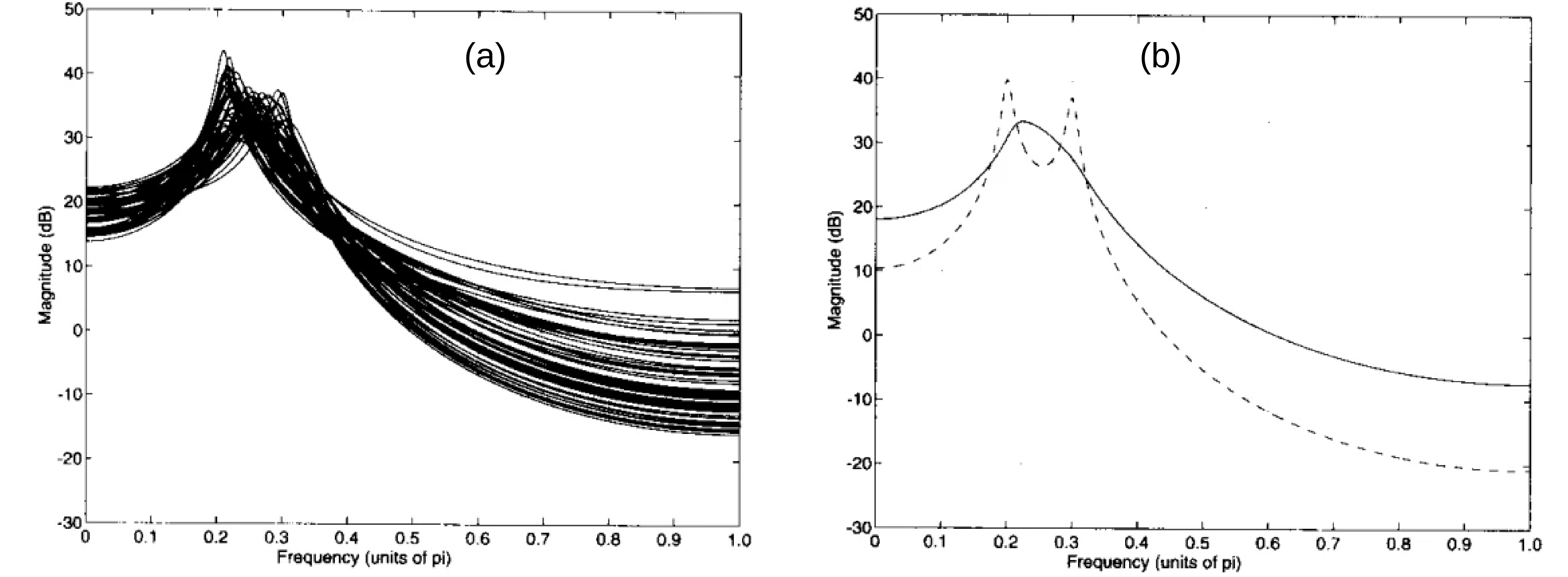


Figure 4: Spectrum estimation of AR(4) process using Autocorrelation method for N=128 (a) Overlay of 50 estimates, (b) average of the estimate (solid line) and true power spectrum (dashed line)

Source: "Parametric Spectrum Estimation, AR (4)-Autocorrelation Method ", gray-sparrow. https://custom-images.strikinglycdn.com/res/hrscywv4p/image/upload/c_limit,fl_lossy,h_9000,w_1200,f_auto,q_auto/24525391/778493_825375.png

AR Spectrum Estimation: Covariance Method

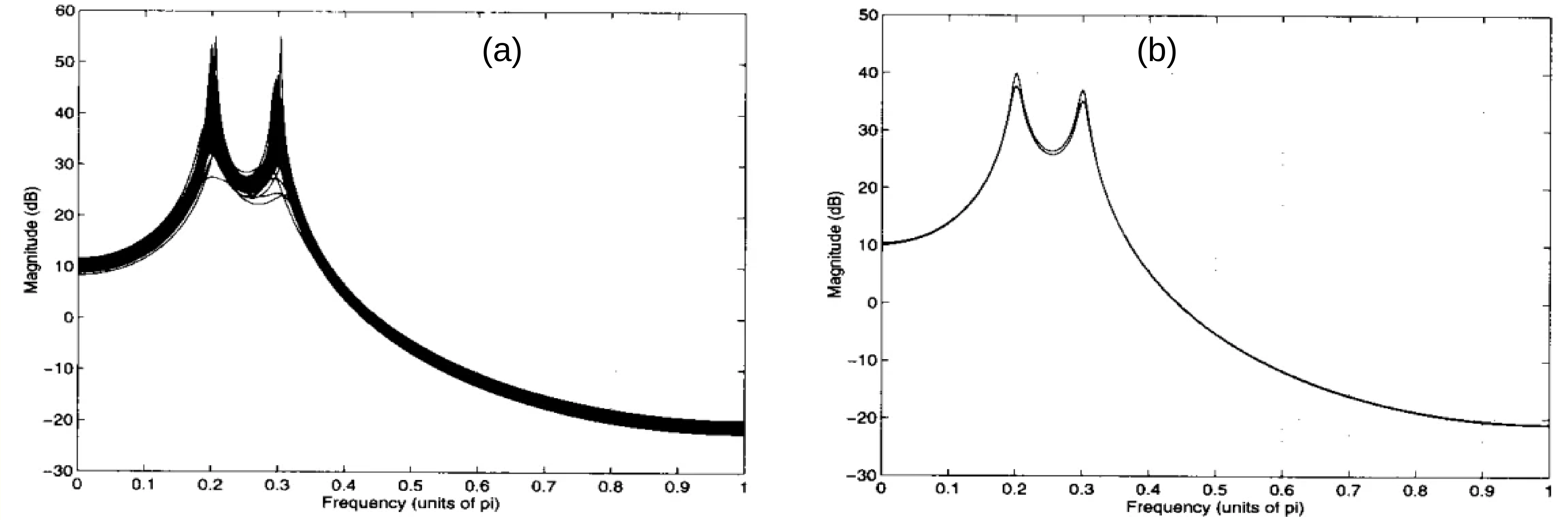


Figure 5: Spectrum estimation of AR(4) process using Covariance method for N=128 (a) Overlay of 50 estimates, (b) average of the estimate (solid line) and true power spectrum (dashed line)

Source: “Parametric Spectrum Estimation, AR (4)-Covariance Method”, gray-sparrow. https://custom-images.strikinglycdn.com/res/hrscywv4p/image/upload/c_limit,fl_lossy,h_9000,w_1200,f_auto,q_auto/24525391/93762_839844.png

AR Spectrum Estimation: Modified Covariance Method

- The modified covariance method is similar to the covariance method in that no window is applied to the data
- The autoregressive parameters in the modified covariance method are found by solving a set of linear equations of the form given in eq(12) with

$$\hat{r}_x(k,l) = \sum_{n=p}^{N-1} \left[x(n-l)x^*(n-k) + x(n-p+l)x^*(n-p+k) \right] \qquad (15)$$

- For the modified covariance method, the autocorrelation matrix is not also Toeplitz
- Compared to other AR spectrum estimation methods, the modified covariance approach provides statistically stable and high-resolution spectrum estimates

AR Spectrum Estimation: Model Order Selection

- A key challenge in applying AR spectrum estimation methods is determining the appropriate model order p for the autoregressive process
- If the chosen model order is too low, the resulting spectrum becomes overly smoothed and exhibits low resolution
- Conversely, if the model order is too high, the spectrum may show artificial or spurious peaks due to the spectral line splitting
- Therefore, it is desirable to have a criterion that can guide the selection of an appropriate model order for a given dataset
- Several criteria have been proposed to find the optimum model order in order to reduce the estimation error
- Those criteria follow the following form

AR Spectrum Estimation: Model Selection

$$C(p) = N \log \varepsilon_p + f(N)p \qquad (16)$$

Where:

N is the data record length

ε_p is the modeling error

$f(N)$ is a constant that depends on N

- Therefore, the objective is to select the value of
- Those criteria follow the following form p that minimizes $C(p)$
- Two widely used criteria's which are aligned with eq(16) are:
  - Akaike Information Criterion (AIC)
  - Minimum Description Length (MDL)

AR Spectrum Estimation: Model Selection

- Akaike Information Criterion (AIC):

$$AIC(p) = N\log\varepsilon_p + 2p \qquad (17)$$

- Minimum Description Length (MDL):

$$MDL(p) = N\log\varepsilon_p + \log N \quad p \qquad (18)$$

- It has been demonstrated that MDL is a consistent estimator of the model order and it converges to the true order as the number of observations N increases
- Two other model selection Criteria Akaike's Final Prediction Error

$$FPE(p) = \varepsilon_P \frac{N+p+1}{N-p-1} \qquad (19)$$

- and Parzen's Criterion Autoregressive Transfer function

$$CAT(p) = \left[\frac{1}{N}\sum_{j=1}^{p}\frac{N-j}{N\varepsilon_j}\right] - \frac{N-p}{N\varepsilon_p} \qquad (20)$$

Moving Average Spectrum Estimation

- Moving average process $x(n)$ can be generated using FIR filter with an input of unit variance white noise $w(n)$ as follows:

$$x(n) = \sum_{k=0}^{q} b_q(k) w(n-k) \qquad (21)$$

- The true power spectrum is also given by:

$$P_x(e^{j\omega}) = \left| \sum_{k=0}^{q} b_q(k) e^{-jk\omega} \right|^2 \qquad (22)$$

- Equivalently the true power spectrum can be written :

$$P_x(e^{j\omega}) = \sum_{k=-q}^{q} r_x(k) e^{-jk\omega} \qquad (23)$$

Moving Average Spectrum Estimation

- The autocorrelation $r_x(k)$ in eq(17) is related to the filter coefficient $b_q(k)$ as:

$$r_x(k) = \sum_{l=0}^{q-k} b_q(l+k) b_q^*(l) \;\; ; \;\; k = 0,1,\ldots,q \qquad (24)$$

Where:

$$r_x(-k) = r_x^*(k) \qquad (25)$$

$$r_x(k) = 0 \;\; ; \;\; |k| > q \qquad (26)$$

- The power spectrum estimate using the moving average method is given by:

$$\hat{P}_{MA}(e^{j\omega}) = \sum_{k=-q}^{q} \hat{r}_x(k) e^{-jk\omega} \qquad (27)$$

Where:

$\hat{r}_x$ is the autocorrelation estimate

Moving Average Spectrum Estimation

- Alternatively, the moving average power spectrum estimate cab performed based on the estimated $b_q(k)$ from $x(n)$ as:

$$\hat{P}_{MA}(e^{j\omega}) = \left| \sum_{k=0}^{q} \hat{b}_q(k) e^{-jk\omega} \right|^2 \qquad (28)$$

Where:

$\hat{b}_q(k)$ is the estimated filter coefficient

Autoregressive Moving Average Spectrum Estimation

- An autoregressive moving average process can be generated by filtering unit variance white noise using the following filter:

$$H(e^{j\omega}) = \frac{B_q(e^{j\omega})}{A_p(e^{j\omega})} = \frac{\sum_{k=0}^{q} b_q(k)e^{-jk\omega}}{1+\sum_{k=1}^{p} a_p(k)e^{-jk\omega}} \tag{29}$$

- The power spectrum of an autoregressive moving average process is given by:

$$P_x(e^{j\omega}) = \frac{\left|\sum_{k=0}^{q} b_q(k)e^{-jk\omega}\right|^2}{\left|1+\sum_{k=1}^{p} a_p(k)e^{-jk\omega}\right|^2} \tag{30}$$

Autoregressive Moving Average Spectrum Estimation

- The autoregressive moving average estimate of the power spectrum is also given by:

$$\hat{P}_{ARMA}(e^{j\omega}) = \frac{\left|\sum_{k=0}^{q} \hat{b}_q(k)e^{-jk\omega}\right|^2}{\left|1+\sum_{k=1}^{p} \hat{a}_p(k)e^{-jk\omega}\right|^2} \qquad (31)$$

Where:

$\hat{b}_q(k)$ and $\hat{a}_p(k)$ are the estimated filter coefficients

Summary

- **Key Deference of Parametric and Non Parametric Spectrum Estimation Methods:**
  - ✓ Parametric methods are model based and include the information about how the process is generated in the power spectrum estimation process
  - ✓ Non-parametric methods do not include the model information in the spectrum estimation
  - ✓ Parametric methods has better accuracy and resolution compared to non parametric ones
- **Major Steps of Parametric Spectrum Estimation**
  - ✓ **Step-1:** Appropriate model selection through prior knowledge or experimentation
  - ✓ **Step-2:** finding the optimal model coefficient
  - ✓ **Step-3:** Estimating the power spectrum using the estimated model parameter and the model equation

Summary

- **Types of Parametric Spectrum Estimation Models**
  - ✓ AR model
  - ✓ MA model
  - ✓ ARMA model

References

[1] Charles W. Therrien,‘’ Discrete Random Signals and Statistical Signal Processing’’, Prentice Hall, Pp.597-598, 1992.

[2] Monson H. Hayes, “Statistical Digital Signal Processing and Modeling’’, John Wiley and sons, Pp.442, 1996.

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Thank You!