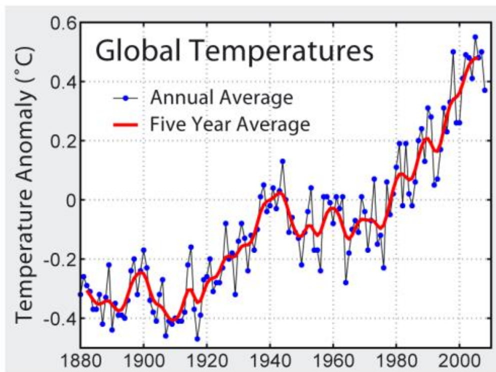
1 Natural climate change: glacial cycles

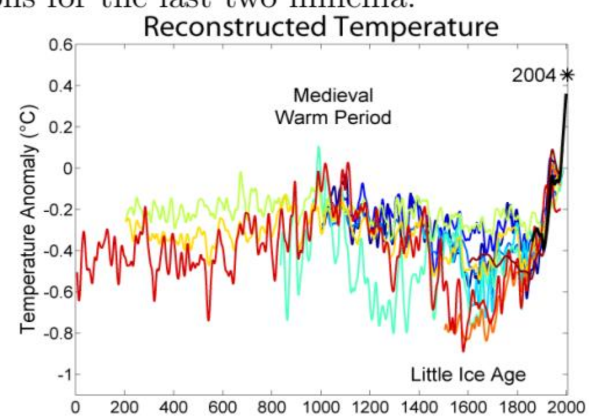
1.1 Climatic cycles

Earth's climate has always fluctuated.

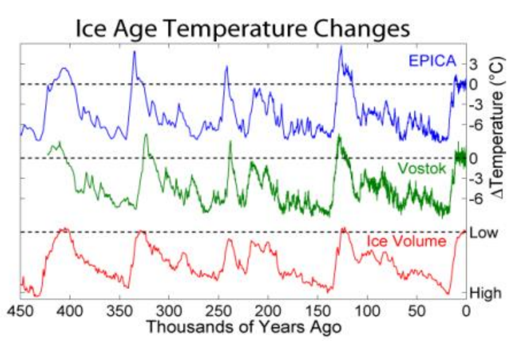
Climate fluctuations since the 19th century:



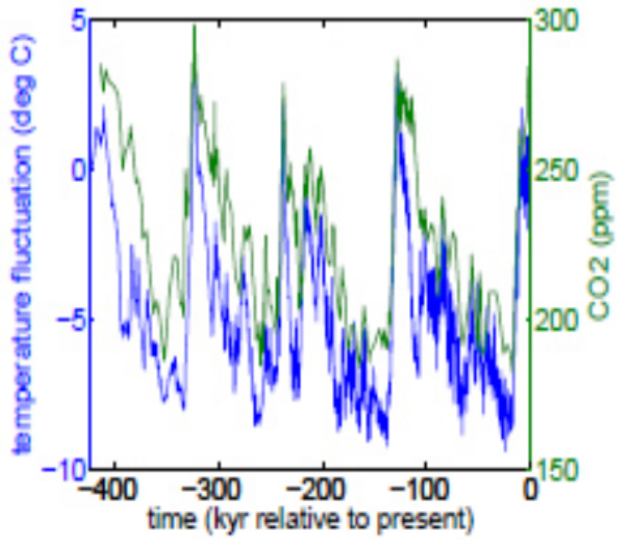
Climate fluctuations for the last two millenia:



Climate fluctuations for the last 450 Kyr exhibit the 100-Kyr periodicity of *glacial cycles*:

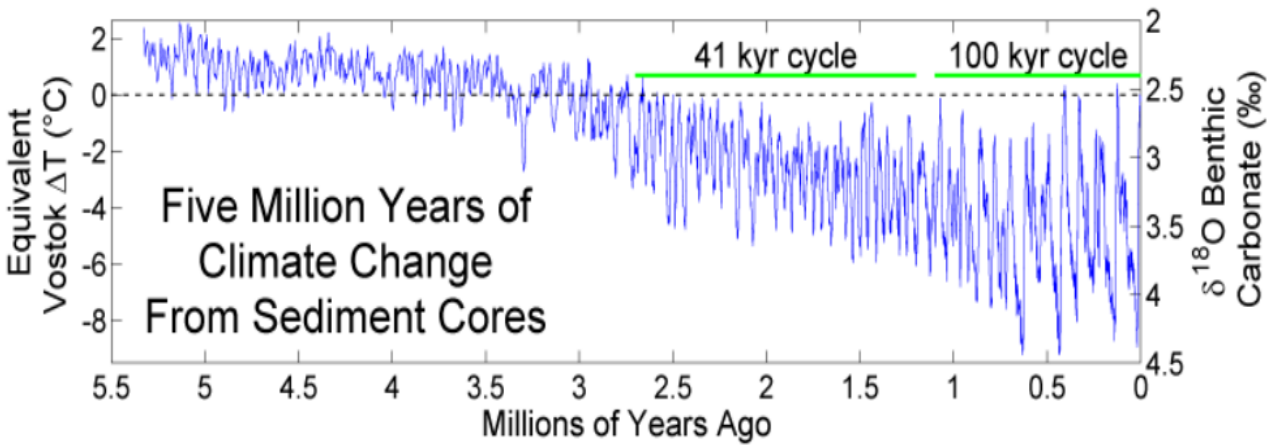


Climate and CO_2 fluctuations for the last 420 Kyr:

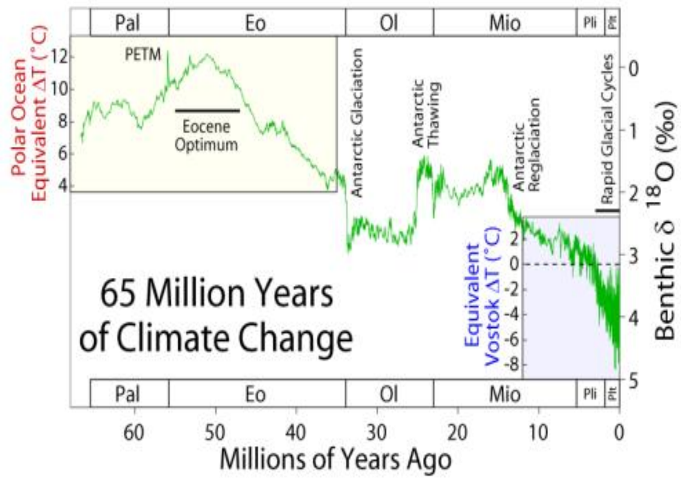


This correlation between pCO_2 and climate was highlighted in Al Gore's film *An Inconvenient Truth*. The covariation of these two signals suggests a strong relation between CO_2 and climate, but its explanation remains one of the great unsolved problems of earth science.

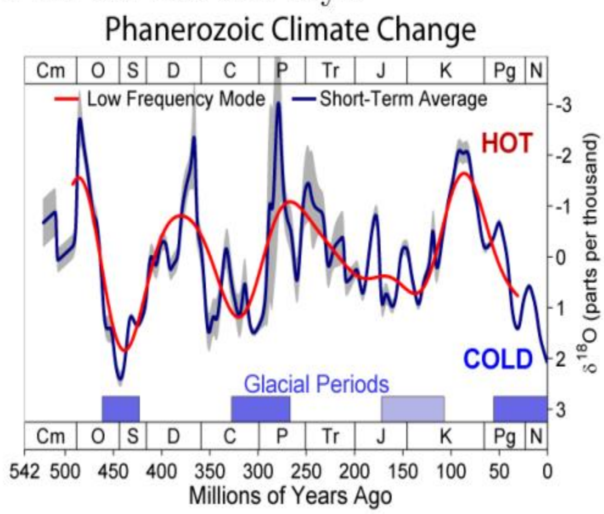
Climate fluctuations for the last 5 Myr show that the 100-Kyr cycle began about 1 Ma, and was preceded by the dominance of a 41-Kyr cycle:



Climate fluctuations for the last 65 Myr:



Climate fluctuations for the last 540 Myr:



1.2 Milankovitch hypothesis: an introduction

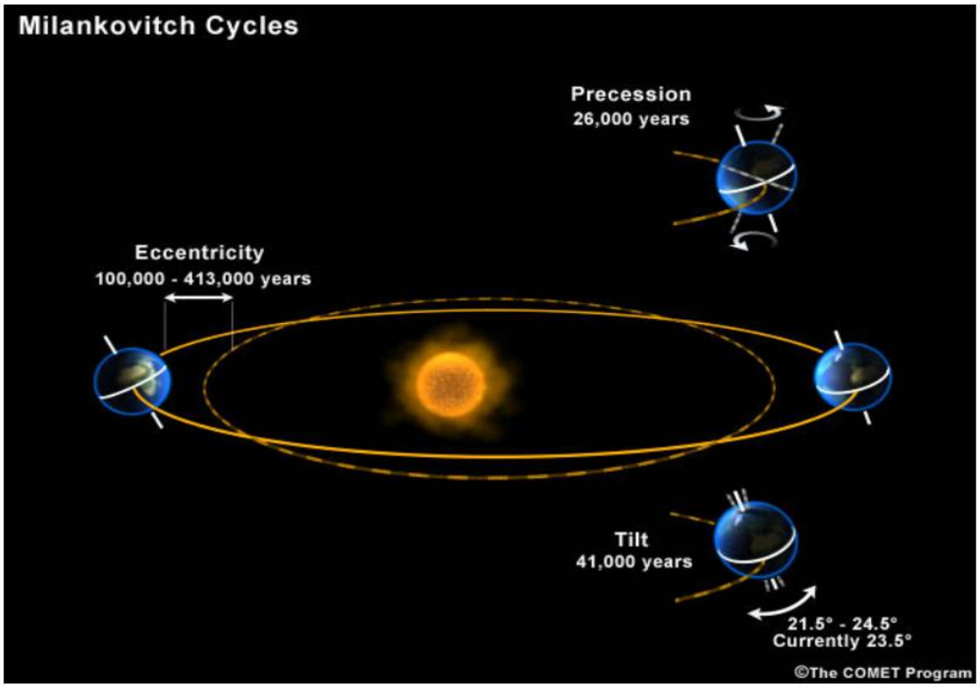
Milutin Milankovitch (1879–1958) proposed that variations in the precession, obliquity, and eccentricity of Earth's orbit are responsible for the glacial cycles.

Similar but less well-developed ideas were proposed in the 19th century by Joseph Adémar and James Croll.

Milankovitch's ideas gained prominence in the 1970s, when evidence of glacial cycles was found in deep sea cores [2].

Let us first take a qualitative look at the three principal orbital parameters.

1.2.1 Precession, obliquity, and eccentricity



- *Precession* is the slow change in the direction of the North Pole.

  Precession results from torques exerted by the Moon and Sun on Earth's equatorial bulge.

  This movement is analogous to that of a tilted top or gyroscope.

  The period of precession is about 25.8 Kyr.

- *Obliquity* is the angle of the *tilt* of the Earth's pole towards the Sun.

  In other words, it is the angle at which the North Pole tilts towards the Sun in summer.

Today the obliquity is 23.5°. Over the last 800 Kyr it has varied between about 22° and 24.5°.

Obliquity varies with a dominant period of 41 Kyr. Its variations are due to torques from Jupiter (because it is large) and other planets.

This rate of change corresponds to 0.13°/Kyr, which means, e.g., that the Tropic of Cancer—the northernmost latitude at which the Sun may appear directly overhead—has moved 1.4 km in the last 100 yr.

- *Eccentricity* quantifies the deviation of Earth's orbit from a perfect circle. Letting

$$\begin{aligned} A &= \text{major axis of the orbit} \\ B &= \text{minor axis} \end{aligned}$$

The eccentricity ε is

$$\varepsilon = \sqrt{1 - \left(\frac{B}{A}\right)^2}.$$

Today

$$A/B = 1.00014 \qquad \text{and} \qquad \varepsilon = 0.017,$$

i.e., the orbit is within 0.014% of being circular. However the distances of the closest and furthest approaches to Sun are at

$$r_{\min} = \frac{A(1-\varepsilon)}{2} \qquad \text{and} \qquad r_{\max} = \frac{A(1+\varepsilon)}{2}$$

so that

$$\frac{r_{\max} - r_{\min}}{A/2} = 2\varepsilon \simeq 3.3\%.$$

We shall show that eccentricity varies with the angular momentum $L = |\vec{L}|$ of Earth's orbit according to

$$\varepsilon^2 = 1 - kL^2$$

where k is approximately constant. L is maximized when the orbit is circular, and any force that increases L decreases the eccentricity.

The rate of change of angular momentum is related to the torque $\vec{\tau}$ on the Earth-Sun system via

$$\frac{\mathrm{d}\vec{L}}{\mathrm{d}t} = \vec{\tau}.$$

Torques on the Earth-Sun system arise from any planet that pulls on the two asymmetrically. The major contributions come from Jupiter (because it is large) and Venus (because it is close).

Eccentricity varies between about 0 and 0.05, with periods of 95, 125, and 400 Kyr.

1.2.2 Insolation

The average flux of solar energy at the top of the Earth's atmosphere is

$$S = 1360 \text{ Watts/m}^2.$$

This is the quantity at normal incidence.

But the flux per unit area—the *insolation*— depends on the tilt of a surface with respect to incoming radiation

Taking the Earth's radius to be R_e, we define

$$W = \text{total solar energy flux received by Earth} = \pi R_e^2 S.$$

But this flux is spread out over an area of size $4\pi R_e^2$. Thus the average daily insolation I is

$$I = S/4 = 340 \text{ W/m}^2.$$

Averaged over a year, this quantity varies neither with precession nor obliquity. It does however vary with eccentricity (due to spherical spreading of the radiation).

This does not mean, however, that precession and obliquity are unimportant.

Indeed, Milankovitch proposed that the main driving force of glacial cycles is summer insolation in the northern hemisphere, since two thirds of the Earth's land area is in the north.

The idea is that summer insolation determines the amount of snow melt, and thus the extent of glaciated surface.

The point is that

- Eccentricity determines total insolation.
- Obliquity and precession determine the *distribution* of insolation.

Note also that the effect of precession depends on how close the Earth comes to the sun, which depends on eccentricity.

We introduce the precession angle

$$\omega_M = \text{angle between spring solstice and perihelion.}$$

(Perihelion is the point where the Earth is closest to the sun.)

The effect of precession on insolation is expressed via the *precession parameter*

$$p = \varepsilon \sin \omega_M.$$

The dominant period of variations in p differ from precession itself because of the moving perihelion—the dominant frequencies correspond to periods of 19, 22, and 24 Kyr.

In what follows we provide a series of physical arguments and elementary calculations so that we may better understand variations in insolation and the orbital parameters that make it vary.

1.3 Precession and obliquity

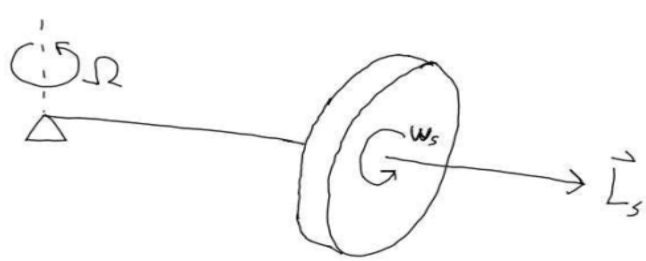
The precession of Earth's axis is analogous to the precession of a gryoscope.

In the following, we show that the uniform precession of a gyroscope is consistent with Newton's laws and the relation between torque and angular momentum (i.e., $\vec{\tau} = \mathrm{d}\vec{L}/\mathrm{d}t$).

We conclude by specifying the analogy with Earth's axial precession.

1.3.1 Gryoscope: horizontal axis

We first suppose that the axis of the gyroscope is horizontal, with one end supported by a free pivot.



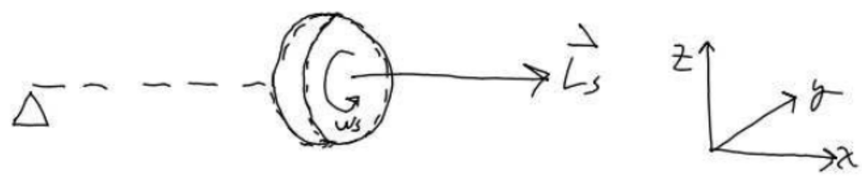
We suppose that the flywheel rotates with angular velocity ω_s.

When the gyroscope is released with a spinning flywheel, it eventually exhibits *uniform precession*, i.e., the axle rotates with constant angular velocity Ω.

Intuitively, we expect that the gyroscope would merely swing vertically about the pivot, like a pendulum. Indeed, this is precisely its behavior when the flywheel does not spin (i.e., $\omega_s = 0$).

But the gyroscope precesses only for large ω_s, i.e., when the flywheel spins rapidly.

In this case virtually all of the gyroscope's angular momentum derives from the spinning flywheel.* Its angular momentum $\vec{L}_s$ is directed along the axle:



The magnitude of $\vec{L}_s$ is

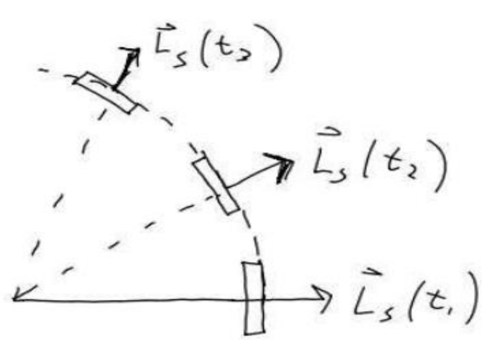
$$|\vec{L}_s| = I_0 \omega_s,$$

where I_0 is the moment of inertia of the flywheel about its axle.

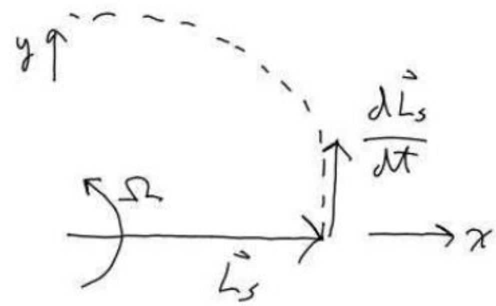
* The small orbital angular momentum is constant for uniform precession.

†Recall the moment of intertia = $\int r^2 \mathrm{d}m$, where r is the distance from the rotation axis and m is mass.

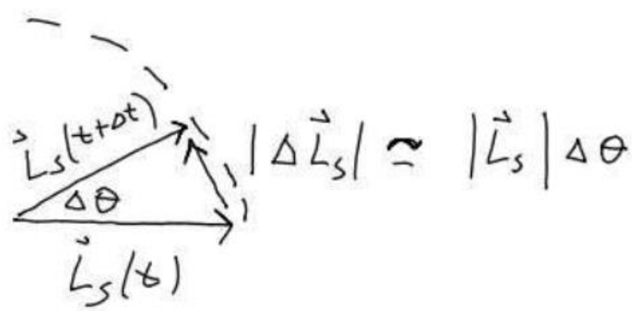
As the gyroscope precesses, $\vec{L}_s$ rotates with it:



Note that $\dfrac{\mathrm{d}\vec{L}_s}{\mathrm{d}t}$ is perpendicular to $\vec{L}_s$:



To determine $\left|\dfrac{\mathrm{d}\vec{L}_s}{\mathrm{d}t}\right|$, we consider small changes in the angular momentum:



Then

$$|\Delta\vec{L}_s| \simeq |\vec{L}_s|\Delta\theta$$

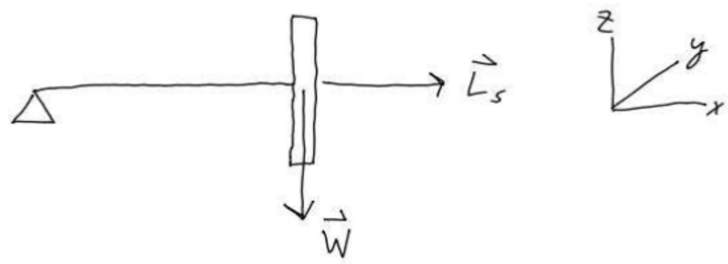
and therefore

$$\begin{aligned}\left|\frac{\mathrm{d}\vec{L}_s}{\mathrm{d}t}\right| &= |\vec{L}_s|\frac{\mathrm{d}\theta}{\mathrm{d}t}\\ &= |\vec{L}_s|\Omega.\end{aligned}$$

Now recall the relation between the torque $\vec{\tau}$ on a body and its angular momentum $\vec{L}$:

$$\vec{\tau} = \frac{\mathrm{d}\vec{L}}{\mathrm{d}t}, \quad \text{where} \quad \vec{\tau} = \vec{r} \times \vec{F}.$$

There must therefore be a torque on the gyroscope. We find that it derives from the weight W of the flywheel:



The torque is directed parallel to $\mathrm{d}\vec{L}_s/\mathrm{d}t$, with magnitude

$$|\vec{\tau}| = \ell W,$$

where ℓ is the distance from the pivot to the flywheel.

Since the torque on the gyroscope is

$$\vec{\tau} = \frac{\mathrm{d}\vec{L}_s}{\mathrm{d}t}$$

we have, by substituting on each side our results from above,

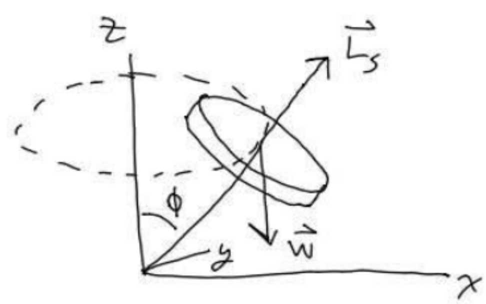
$$\ell W = |\vec{L}_s|\Omega$$

and therefore the angular velocity of precession is

$$\Omega = \frac{\ell W}{|\vec{L}_s|} = \frac{\ell W}{I_0 \omega_s}.$$

1.3.2 Gyroscope: tilted axis

Now imagine that the axis of the gyroscope is not horizontal but is instead tilted at an angle ϕ with the vertical:



The vertical (z) component of $\vec{L}_s$ is constant.

The horizontal component varies, but always has magnitude

$$|\vec{L}_s|_{\text{horiz}} = |\vec{L}_s| \sin\phi.$$

Since only the horizontal component contributes to $\mathrm{d}\vec{L}_s/\mathrm{d}t$, we have, reasoning as above,

$$\left|\frac{\mathrm{d}\vec{L}_s}{\mathrm{d}t}\right| = \Omega|\vec{L}_s| \sin\phi.$$

The torque arising from gravity (i.e., $\vec{r} \times \vec{W}$) is again horizontal, but now with magnitude

$$|\vec{\tau}| = \ell W \sin\phi.$$

Using once again that $\vec{\tau} = \mathrm{d}\vec{L}_s/\mathrm{d}t$, we combine the previous two relations to obtain

$$\ell W \sin\phi = \Omega|\vec{L}_s| \sin\phi.$$

We find that the precessional velocity is once again

$$\Omega = \frac{\ell W}{|\vec{L}_s|},$$

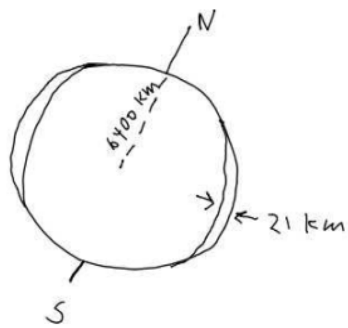
independent of the angle ϕ.

1.3.3 Planetary precession

We now address the precession of Earth's rotation axis.

If the Earth were perfectly spherical and its only interaction were with the Sun, then there would be no torques on it and its angular momentum would always point in the same direction.

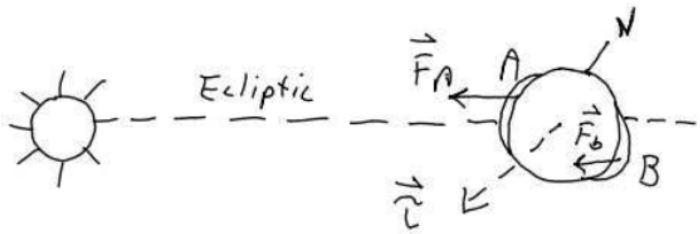
However a torque arises because of the non-spherical shape of the Earth: the mean equatorial radius is about 21 km greater than the polar radius (about 6400 km):



The torque exists because

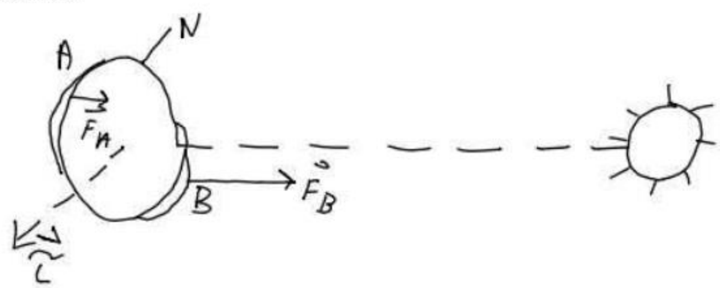
- the Earth's rotation axis is tilted with respect to the orbital plane (the "ecliptic"), by about 23.5°; and
- the Sun pulls asymmetrically on the equatorial bulge.

During the northern hemisphere winter, the bulge above the ecliptic is attracted more strongly to the Sun (F_A) than the bulge below the ecliptic (F_B):



There is thus a counterclockwise torque, out of the plane of the figure.

In summer, B is attracted more strongly to the Sun, but the torque remains in the same direction:



In spring and fall, on the other hand, the torque is zero.

Thus the average torque is in the plane perpendicular to the spin axis, in the plane of the ecliptic.

The moon has the same effect (with about twice the torque).

Consequently the Earth's rotational axis precesses.

The period of the Earth's precession is about 26,000 yr.

Thus, while the Earth's spin axis presently points towards Polaris, this "North Star" will be $2 \times 23.5° = 47°$ off-axis in 13,000 yr.

1.3.4 Obliquity

Whereas precession is the rotation of Earth's spin axis, obliquity is the angle of the axis.

From the preceding discussion, we know that the vertical component of the angular momentum $\vec{L}_s$ due to spin is constant.

However that will only be the case if there are no torques on the Earth outside the Earth-Sun interaction.

We can thus identify changes in Earth's obliquity with torques applied to it.

Aside from the moon, these torques can also come from interactions with other planets, especially Jupiter because it is large, and Venus because it is close, as we discuss at the end of Section 1.4.

Earth's obliquity varies by about $\pm 1°$, with a period of about 41 Kyr.

1.4 Eccentricity

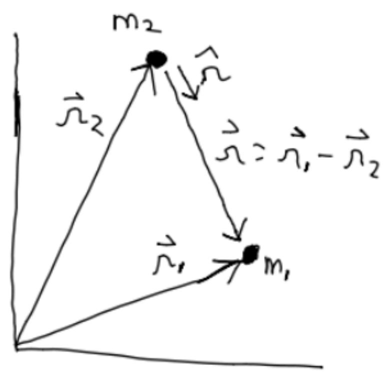
We next analyze the eccentricity of Earth's orbit.

We first examine the problem of *central force motion*, and show that planetary orbits are elliptical.

In doing so, we derive an expression for eccentricity, emphasizing how changes in the Earth's angular momentum can change the eccentricity of its orbit.

1.4.1 Central force motion as a one-body problem

Consider two particles interacting via a force $f(r)$, with masses m_1, m_2 and position vectors $\vec{r}_1$, $\vec{r}_2$.



We define

$$\vec{r} = \vec{r}_1 - \vec{r}_2 \tag{1}$$
$$r = |\vec{r}| = |\vec{r}_1 - \vec{r}_2| \tag{2}$$

For an attractive force $f(r) < 0$, we have the equations of motion

$$m_1\ddot{\vec{r}}_1 = f(r)\hat{r} \tag{3}$$
$$m_2\ddot{\vec{r}}_2 = -f(r)\hat{r}. \tag{4}$$

We simplify this system by noting that the center of mass is located at

$$\vec{R} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2}{m_1 + m_2}. \tag{5}$$

Since there are no external forces on the center of mass,

$$\ddot{\vec{R}} = 0$$

and therefore

$$\vec{R}(t) = \vec{R}_0 + \vec{V}t$$

Taking the origin at the center of mass,

$$\vec{R}_0 = 0 \quad \text{and} \quad \vec{V} = 0.$$

We next seek an equation of motion for $\vec{r} = \vec{r}_1 - \vec{r}_2$. We rewrite equations (3) and (4) as

$$\begin{aligned} \ddot{\vec{r}}_1 &= \frac{f(r)\hat{r}}{m_1} \\ \ddot{\vec{r}}_2 &= \frac{-f(r)\hat{r}}{m_2}, \end{aligned}$$

Subtracting the latter from the former, we have

$$\ddot{\vec{r}}_1 - \ddot{\vec{r}}_2 = \left(\frac{1}{m_1} + \frac{1}{m_2}\right) f(r)\hat{r}$$

We rewrite this expression as

$$\mu\ddot{\vec{r}} = f(r)\hat{r} \tag{6}$$

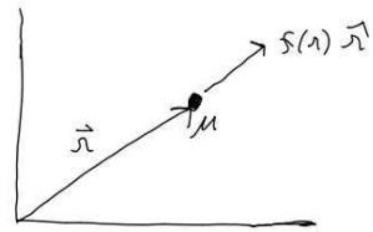
where

$$\mu = \frac{m_1 m_2}{m_1 + m_2} \tag{7}$$

is the *reduced mass.*

We have thus reduced the two particle problem to a one-particle problem, described by equation of motion (6) for a particle of mass μ subjected to a force $f(r)\hat{r}$:

We have thus reduced the two particle problem to a one-particle problem, described by equation of motion (6) for a particle of mass μ subjected to a force $f(r)\hat{r}$:

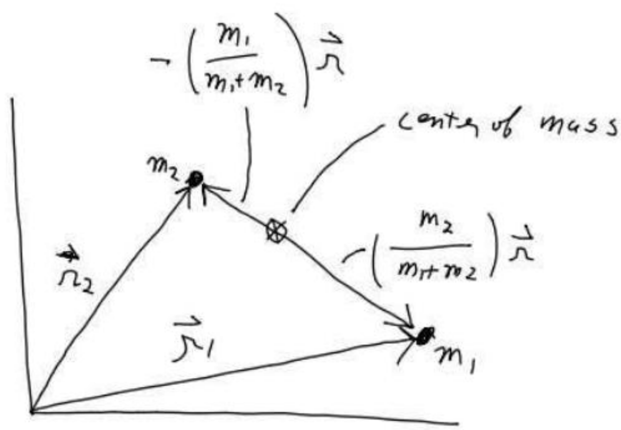


The essential problem is to solve (6) for $\vec{r}(t)$. Then, using (1) and (5), we find the original position vectors

$$\vec{r}_1 = \vec{R} + \left(\frac{m_2}{m_1 + m_2}\right)\vec{r} \tag{8}$$

$$\vec{r}_2 = \vec{R} - \left(\frac{m_1}{m_1 + m_2}\right)\vec{r} \tag{9}$$

where the second term on the RHS of each relation above indicates the position vector relative to the center of mass:



References

[1] Muller, R. A. & Macdonald, G. J. *Ice Ages and Astronomical Causes* (Springer, New York, 2000).

[2] Hayes, J. D., Imbrie, J. & Shackleton, N. J. Variations in the earth's orbit: Pacemaker of the ice ages. *Science* **194**, 1121–1132 (1976).

[3] Kleppner, D. & Kolenkow, R. J. *An Introduction to Mechanics* (Cambridge U. Press, New York, 2010).

[4] Danby, J. *Fundamentals of Celestial Mechanics, Second edition* (Willman-Bell, Richmond, Virginia, 1988).

[5] Moulton, F. R. *An Introduction to Celestial Mechanics, Second revised edition* (Dover Publications, New York, 1970).

[6] Berger, A. & Loutre, M. F. Precession, eccentricity, obliquity, insolation, and paleoclimates. In Duplessy, J.-C. & Spyridakis, M.-T. (eds.) *Long-Term Climatic Variations*, vol. 122 of *NATO ASI Series*, 107–151 (Springer-Verlag, Heidelberg, 1994).