1.6.2 Heating of a semi-infinite medium

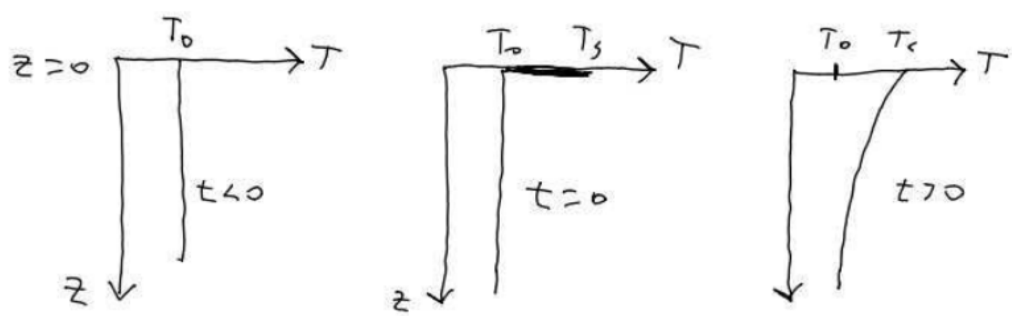
We next consider a classic problem. Suppose that the temperature is a constant (say, $T = T_0$) for all $z > 0$, but at time $t = 0$, the surface $z = 0$ is held at a higher or lower temperature T_s.

How does the temperature $T(z, t)$ of this half-space, or semi-infinite medium, change in time and space?

In other words, we seek the time-dependent solution of the heat equation subject to the conditions

$$\begin{aligned} T &= T_0, \quad z > 0, \quad t = 0 \\ T &= T_s, \quad z = 0, \quad t > 0 \\ T &\to T_0, \quad z \to \infty, \quad t > 0. \end{aligned}$$

Conceptually, we envision the transfer of heat from the surface $z = 0$ inwards.



(Note that in the seafloor problem, the half-space will be cooled from above, so that $T_s < T_0$, but the mathematical development is unchanged.)

To make matters simpler, we consider only the difference $T - T_0$ of temperature with respect to the amount $T_s - T_0$ by which the surface is heated. We therefore define the dimensionless variable

$$\theta = \frac{T - T_0}{T_s - T_0}.$$

The diffusion equation (2) then becomes

$$\frac{\partial \theta}{\partial t} = D\frac{\partial^2 \theta}{\partial z^2} \tag{3}$$

and our boundary conditions are now

$$\begin{aligned}\theta(z,0) &= 0\\ \theta(0,t) &= 1\\ \theta(\infty,t) &= 0.\end{aligned}$$

Obviously θ varies with both time and space. However, because no characteristic length scale exists (i.e., the medium is semi-infinite, and the heating is at the boundary), the only quantity other than z with dimensions of length is

$$(Dt)^{1/2}.$$

We therefore suspect that

$$\theta(z,t) = \theta(\eta),$$

i.e., that θ is a function of the single dimensionless variable

$$\eta = \frac{z}{2\sqrt{Dt}},$$

where the factor of two is chosen for convenience.

η is called a *similarity variable*, because solutions at different points z and different times t are equivalent so long as η is unchanged. The solutions are then said to be *self-similar*.

As a practical matter, expressing the diffusion equation in terms of $\theta(\eta)$ rather than $\theta(z,t)$ allows us to transform a partial-differential equation to an ordinary differential equation. This is a classic trick in such problems, which in this case is known as the *Boltzmann transformation*.

We proceed as follows. Using the chain rule, we transform the LHS of the diffusion equation:

$$\begin{aligned}\frac{\partial \theta}{\partial t} &= \frac{\mathrm{d}\theta}{\mathrm{d}\eta}\frac{\partial \eta}{\partial t}\\ &= \frac{\mathrm{d}\theta}{\mathrm{d}\eta}\left(-\frac{1}{4t}\frac{z}{\sqrt{Dt}}\right)\\ &= -\frac{\eta}{2t}\frac{\mathrm{d}\theta}{\mathrm{d}\eta}.\end{aligned}$$

To obtain the RHS, we first write

$$\begin{aligned}\frac{\partial \theta}{\partial z} &= \frac{\mathrm{d}\theta}{\mathrm{d}\eta}\frac{\partial \eta}{\partial z} \\ &= \frac{1}{2\sqrt{Dt}}\frac{\mathrm{d}\theta}{\mathrm{d}\eta}.\end{aligned}$$

Using this result and again invoking the chain rule,

$$\begin{aligned}\frac{\partial}{\partial z}\frac{\partial \theta}{\partial z} &= \frac{\partial}{\partial z}\left(\frac{1}{2\sqrt{Dt}}\frac{\mathrm{d}}{\mathrm{d}\eta}\theta[\eta(z,t)]\right) \\ &= \frac{1}{2\sqrt{Dt}}\frac{\partial \eta}{\partial z}\frac{\mathrm{d}^2\theta}{\mathrm{d}\eta^2} \\ &= \frac{1}{4Dt}\frac{\mathrm{d}^2\theta}{\mathrm{d}\eta^2}.\end{aligned}$$

Substitution of these results into the diffusion equation (3) then yields

$$-\frac{\eta}{2t}\frac{\mathrm{d}\theta}{\mathrm{d}\eta} = D\frac{1}{4Dt}\frac{\mathrm{d}^2\theta}{\mathrm{d}\eta^2}$$

or more simply

$$-\eta\frac{\mathrm{d}\theta}{\mathrm{d}\eta} = \frac{1}{2}\frac{\mathrm{d}^2\theta}{\mathrm{d}\eta^2} \tag{4}$$

Note that all terms are now dimensionless.

To obtain the boundary conditions, note that

$$\begin{aligned} z = 0 &\rightarrow \eta = 0, \\ z = \infty &\rightarrow \eta = \infty, \\ t = 0 &\rightarrow \eta = \infty. \end{aligned}$$

The new boundary conditions are then

$$\begin{aligned}\theta(0) &= 1 \\ \theta(\infty) &= 0\end{aligned}$$

To solve equation (4) with these boundary conditions, define

$$\phi = \frac{\mathrm{d}\theta}{\mathrm{d}\eta} \tag{5}$$

and substitute into (4) to obtain

$$-\eta\phi = \frac{1}{2}\frac{\mathrm{d}\phi}{\mathrm{d}\eta}$$

or, equivalently,

$$-2\eta\,\mathrm{d}\eta = \frac{\mathrm{d}\phi}{\phi}.$$

Integrating both sides, we obtain

$$-\eta^2 = \ln\phi - \ln C_1,$$

where $-\ln C_1$ is an integration constant. We thus have that

$$\phi = C_1 e^{-\eta^2}.$$

Combining this result with equation (5), we find

$$\frac{\mathrm{d}\theta}{\mathrm{d}\eta} = C_1 e^{-\eta^2}.$$

which upon integration becomes

$$\theta(\eta) = C_1 \int_0^\eta e^{-\eta'^2}\mathrm{d}\eta' + C_2. \tag{6}$$

Using the boundary condition $\theta(0) = 1$, we find that the integration constant

$$C_2 = 1.$$

Thus the second b.c., $\theta(\infty) = 0$, requires that

$$0 = C_1 \quad {}^{\infty}_{0} \quad e^{-\eta'^2}\mathrm{d}\eta' + 1$$

and therefore

$$\begin{aligned} \frac{-1}{C_1} &= \int_0^\infty e^{-\eta'^2}\mathrm{d}\eta' \\ &= \frac{\sqrt{\pi}}{2}, \end{aligned}$$

which may be found in a table of integrals. Inserting the constants C_1 and C_2 into (6), we find

$$\theta(\eta) = \frac{-2}{\sqrt{\pi}} \int_0^\eta e^{-\eta'^2}\mathrm{d}\eta' + 1,$$

or more succinctly

$$\theta = 1 - \mathrm{erf}(\eta)$$

where the *error function*

$$\mathrm{erf}(\eta) = \frac{2}{\sqrt{\pi}} \int_0^{\eta} e^{-\eta'^2} \mathrm{d}\eta'.$$

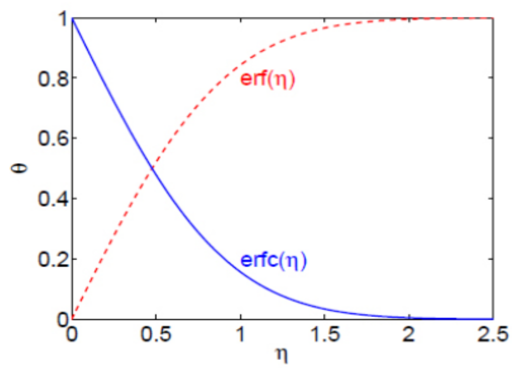
Not only does erf occur commonly in mathematics, physics, and probability theory, but so too does the *complementary error function*

$$\mathrm{erfc}(\eta) = 1 - \mathrm{erf}(\eta).$$

Using this definition, the rescaled temperature θ is simply

$$\theta = \mathrm{erfc}(\eta),$$

which is plotted below as the solid blue line:



Reverting to dimensional variables, we have

$$\frac{T - T_0}{T_s - T_0} = \mathrm{erfc}\left(\frac{z}{2\sqrt{Dt}}\right) \tag{7}$$

Looking at the plot, we see that there is a characteristic *diffusion length*

$$\eta \simeq 1 \quad \Rightarrow \quad z \simeq \sqrt{Dt}$$

over which heat extends from the boundary into the medium.

The essential result is that the thickness of this *thermal boundary layer* grows like $t^{1/2}$, which is typical in problems of diffusion.

1.6.3 Heat flux

The heat flux at $z = 0$ is straightforwardly obtained by computing the flux $j(0)$ from Fourier's law:

$$\begin{aligned} j(0) &= -k\left.\frac{\partial T}{\partial z}\right|_{z=0} \\ &= -k(T_s - T_0)\frac{\partial}{\partial z}\left.\operatorname{erfc}\left(\frac{z}{2\sqrt{Dt}}\right)\right|_{z=0} \\ &= k(T_s - T_0)\frac{\partial}{\partial z}\left.\operatorname{erf}\left(\frac{z}{2\sqrt{Dt}}\right)\right|_{z=0} \end{aligned}$$

Noting that $\frac{\partial}{\partial z}\operatorname{erf}[\eta(z,t)] = \frac{\partial \eta}{\partial z}\frac{\mathrm{d}}{\mathrm{d}\eta}\operatorname{erf}(\eta)$,

$$\begin{aligned} j(0) &= \frac{k(T_s - T_0)}{2\sqrt{Dt}}\frac{\mathrm{d}}{\mathrm{d}\eta}\left.\operatorname{erf}(\eta)\right|_{\eta=0} \\ &= \frac{k(T_s - T_0)}{2\sqrt{Dt}}\frac{2}{\sqrt{\pi}}\left. e^{-\eta^2}\right|_{\eta=0} \\ &= \frac{k(T_s - T_0)}{\sqrt{D\pi}}\, t^{-1/2}. \end{aligned}$$

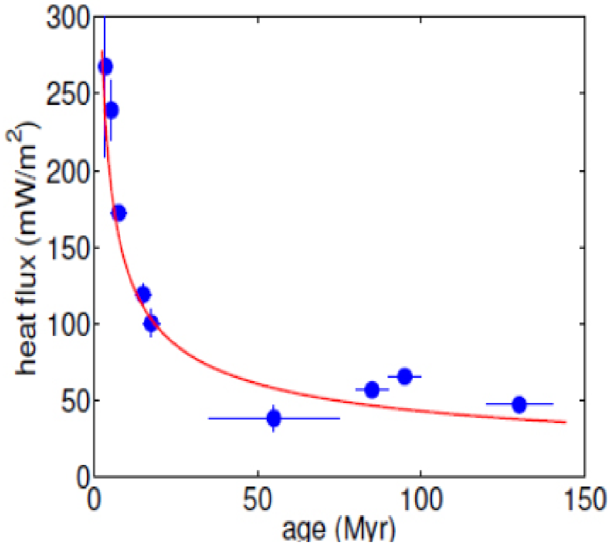
The heat flux thus decreases like $t^{-1/2}$. (The singularity at $t = 0$ is merely the consequence of the instantaneous imposition of a point source.)

We now return to the problem of the generation of new seafloor at mid-ocean ridges.

New seafloor is created at a high temperature and is cooled from above as it ages. Typical dimensional quantities are

$$\begin{aligned} T_0 - T_s &= 1300\ {}^\circ\mathrm{K} \\ D &= 1\ \mathrm{mm}^2\ \mathrm{s}^{-1} \\ k &= 3.3\ \mathrm{W\ m}^{-1}\ {}^\circ\mathrm{K}^{-1} \end{aligned}$$

Measurements of heat flux at the seafloor compare reasonably well with predictions based on its age t



1.6.4 Seafloor topography

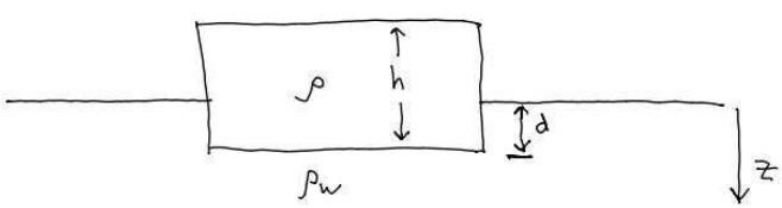
As the new seafloor cools, it becomes more dense and therefore less buoyant with respect to the more dense, underlying mantle. From this observation we can predict the depth of the seafloor as a function of its age.

The key point is that the mantle acts like a viscous fluid and the seafloor (i.e., the oceanic lithosphere) floats on it, but at a height determined by its density.

To quantify this, recall *Archimedes principle*:

> Any floating object displaces its own weight of fluid.

Consider, for example, a wooden block of vertical extent h in a bathtub. The block is partially immersed to a depth $d < h$:



Take ρ_w to be the density of the water and ρ the density of the block. From Archimedes' principle, we have the *hydrostatic equilibrium* in which

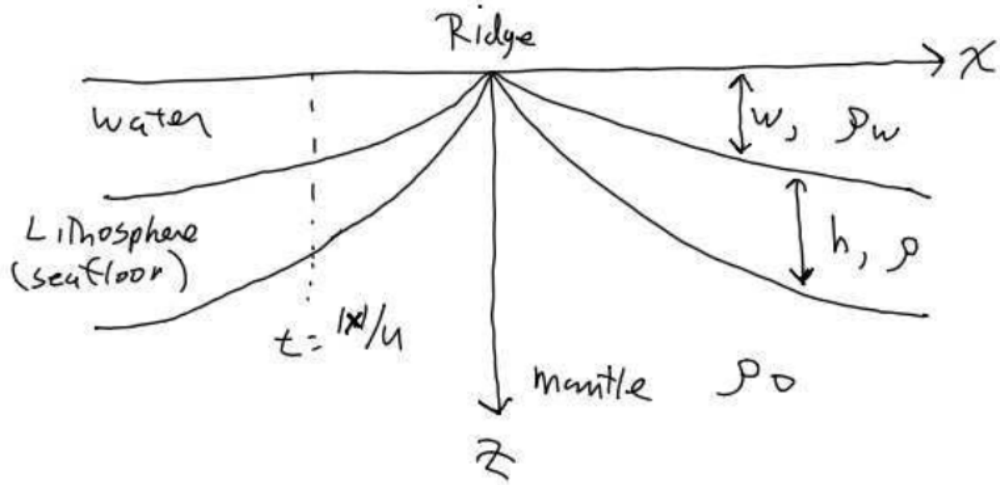
$$\rho h = \rho_w d.$$

Thus the height $h - d$ of the block relative to the water level is

$$\begin{aligned} h - d &= h - \frac{\rho}{\rho_w} h \\ &= h\left(1 - \frac{\rho}{\rho_w}\right). \end{aligned}$$

To apply this reasoning to the seafloor, we note that the combined weight of ocean + seafloor must remain the same as the seafloor cools and becomes more dense.

We consider depth $z(x)$ relative to the height of the mid-ocean ridge, and seek the specific depth $w(x)$ of the seafloor. (x is the distance from the ridge, and w is analogous to $h - d$ above.)



The seafloor (i.e., oceanic lithosphere) thickens as it cools, because underlying hot mantle material is converted to lithosphere.

Define

$$h(x) = \text{seafloor thickness a distance } x \text{ from the ridge.}$$

Then the weight of any column (of ocean + lithosphere) a depth h beneath the seafloor is simply the weight of the water + the weight of the lithospheric part of the column.

Let

$\rho(x, z)$ = density of lithosphere a distance x from the ridge at depth z.

From Archimedes, we know that the integrated mass in any column to a depth $w + h$ must equal the mass computed using the density ρ_0 of the mantle over the same vertical extent. Thus

$$\rho_w w + \int_0^h \rho \, \mathrm{d}z = \rho_0(w + h)$$

or

$$w(\rho_w - \rho_0) + \int_0^h (\rho - \rho_0) \mathrm{d}z = 0 \tag{8}$$

This is known as *isostatic equilibrium*. The first term is negative (mantle rock is denser than water). The second term is positive, since, as it cools, the density of the lithosphere becomes heavier than the mantle density.

Specifically, assume that the density ρ of the seafloor increases with respect to the mantle density ρ_0 according to

$$\rho - \rho_0 = -\rho_0 \alpha (T - T_0),$$

where T_0 is the temperature of new seafloor before it starts to cool, and α is the coefficient of thermal expansion.

Inserting equation (7) for $T - T_0$ we obtain

$$\rho - \rho_0 = -\rho_0 \alpha (T_s - T_0) \operatorname{erfc}\left(\frac{z}{2\sqrt{Dt}}\right) \tag{9}$$

where T_s is the temperature of the seawater above the seafloor and we take

$$\rho(x, z) \to \rho(t, z), \qquad t = |x|/u$$

where t is the age of the seafloor and u is the spreading rate.

We insert (9) into the isostatic equilibrium (8) to obtain

$$w(t)(\rho_0 - \rho_w) = \rho_0 \alpha (T_0 - T_s) \int_0^\infty \operatorname{erfc}\left(\frac{z}{2\sqrt{Dt}}\right) \mathrm{d}z \qquad (10)$$

where $w(t)$ is now the depth of the seafloor with age t and the integral now extends to $z = \infty$.

Substituting the similarity variable $\eta = z/(2\sqrt{Dt})$ into the integral, we have

$$\begin{aligned} \int_0^\infty \operatorname{erfc}\left(\frac{z}{2\sqrt{Dt}}\right) \mathrm{d}z &= \int_0^\infty \operatorname{erfc}(\eta) \left|\frac{\mathrm{d}z}{\mathrm{d}\eta}\right| \mathrm{d}\eta \\ &= 2\sqrt{Dt} \int_0^\infty \operatorname{erfc}(\eta) \mathrm{d}\eta \\ &= \frac{2\sqrt{Dt}}{\sqrt{\pi}}. \end{aligned}$$

where we have used $\int_0^\infty \operatorname{erfc}(\eta)\mathrm{d}\eta = 1/\sqrt{\pi}$.

Finally, we substitute this expression into (10) to obtain

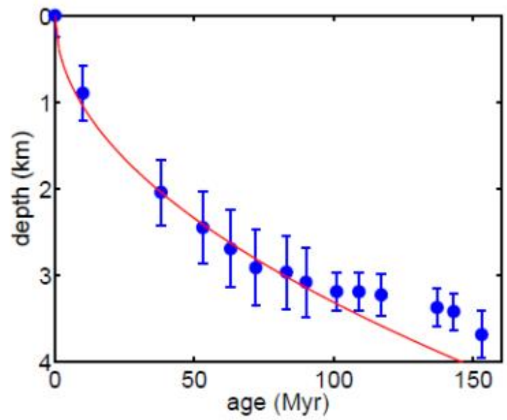
$$w(t) = \frac{2D^{1/2}\rho_0 \alpha (T_0 - T_s)}{\pi^{1/2}(\rho_0 - \rho_w)} t^{1/2}.$$

Note that we have once again found a diffusive scaling in which a length varies like $t^{1/2}$.

We test this prediction by comparison with seafloor bathymetry from the North Atlantic, using data from Ref. [6]. We use the same values of $T_s - T_0$ and D we used for the heat flux in Section 1.6.3, supplemented by

$$\begin{aligned} \rho_0 &= 3300 \text{ kg m}^{-3} \\ \rho_w &= 1000 \text{ kg m}^{-3} \\ \alpha &= 2.8 \times 10^{-5}\,{}^\circ\mathrm{K}^{-1} \end{aligned}$$

The agreement for ages up to about 80 Myr is excellent:



Possibly the departure from theory at old ages is due to some secondary heat transfer from the mantle due, e.g., to convection.

1.6.5 Spreading rate and CO_2 production

The foregoing analysis amounts to an excellent confirmation of seafloor spreading.

We then ask: what controls the rate at which CO_2 is "produced" by degassing of magma at mid-ocean ridges?

A reasonable assumption is that the CO_2 production rate is proportional to the rate of seafloor production, which we express in terms of the spreading rate u:

$$\begin{aligned} CO_2 \text{ production rate} \propto u &= \text{spreading rate} \\ &\simeq 2\text{–}10 \text{ cm yr}^{-1}. \end{aligned}$$

We found above, in Section 1.6.3, that the heat flux j from the seafloor decreases with its age t like

$$j \propto t^{-1/2}.$$

Assuming that seafloor age is uniformly distributed, the mean (spatially averaged) heat flux $\bar{j}$ then scales like

$$\begin{aligned} \bar{j} &\propto \frac{1}{t_{\max}} \int_0^{t_{\max}} t^{-1/2} \mathrm{d}t \\ &\propto t_{\max}^{-1/2}, \end{aligned}$$

where $t_{\max}$ is the typical age at which seafloor is consumed (i.e., subducted).

Taking L to be a characteristic distance between ridges and subduction zones, the age of subducting materal then scales with the spreading rate like

$$t_{\max} \sim L/u.$$

The mean heat flux therefore scales like

$$\bar{j} \sim \frac{1}{(L/u)^{1/2}} \propto u^{1/2}. \tag{11}$$

Invoking the assumption that CO_2 production scales like the spreading rate u, we then have that the

$$CO_2 \text{ production rate} \propto \bar{j}^2.$$

The quadratic dependence of CO_2 production and spreading rate on the heat flux results from diffusive scaling.

This nonlinear amplification informs our knowledge of past environments: because the heat flux from the early Earth would have been about 2–3 times greater than today, CO_2 production from "volcanism" would have been about 4–9 times greater

1.6.6 Kelvin's estimate of the age of the Earth

The foregoing amounts to the "new view of the Earth" that first developed during the 1960s. One way to appreciate how huge such changes were is to consider Kelvin's famous 19th century estimate of the age of the Earth.

Kelvin's reasoning was delightfully simple, making the following assumptions:

- The Earth began its geological history at a uniform temperature of about $T = 4000$ °C.
- The earth then cooled, due to a constant $T \simeq 0$ °C boundary condition at the surface.

Reasoning as in Section 1.6.3, he found that

$$\text{heat flux} \propto \frac{1}{\sqrt{\text{age}}} \tag{12}$$

and used estimates of the heat flux in Edinburgh and appropriate dimensional constants to find that the Earth's age was about 20 Myr.

This not only threatened the biblical version of Earth history, but also the view of the geologist Lyell, for whom the Earth was unchanging throughout essentially infinite time (a concept known as "uniformitarianism").

The eventual knowledge of radioactivity and the advent of age dating proved Kelvin wrong: the Earth is about 4.5 billion years old, about 100 times older than Kelvin's estimate.

What was wrong with Kelvin's theory?

There are two problems:

- Kelvin was unaware of the heating due to radioactive decay, which provides a prolonged source.
- He ignored thermal convection.

The latter problem turns out to be the more serious deficiency, because convection greatly increases the observed heat flow (see equation (11)), and therefore, via (12), it can easily lead to a serious underestimate of age.

1.7 Summary

Thermal convection is the "engine" that drives plate tectonics and volcanism.

It turns out that volcanism is, over the long-term, responsible for the CO_2 in the atmosphere, and thus the source of carbon that is fixed by plants.

Thus volcanism may be said to sustain life.

Without it, there probably would be no CO_2 in the atmosphere, and therefore we wouldn't be around to discuss it...

References

[1] Morner, N. A. & Etiope, G. Carbon degassing from the lithosphere. *Global and Planetary Change* **33**, 185–203 (2002).

[2] Bergé, P., Pomeau, Y. & Vidal, C. *Order within Chaos: Towards a Deterministic Approach to Turbulence* (John Wiley and Sons, New York, 1984).

[3] Hards, V. Volcanic contributions to the global carbon cycle. Ocasional Publication 10, British Geological Society (2005).

[4] Turcotte, D. L. & Schubert, G. *Geodynamics: Applications of Continuum Physics to Geological Problems* (Wiley, New York, 1982).

[5] Sclater, J. G., Crowe, J. & Anderson, R. N. On the reliability of oceanic heat flow averages. *Journal of Geophysical Research* **81**, 2997–3006 (1976).

[6] Parsons, B. & Sclater, J. G. Analysis of variation of ocean-floor bathymetry and heat-flow with age. *Journal of Geophysical Research* **82**, 803–827 (1977).

[7] Sleep, N. H. & Zahnle, K. Carbon dioxide cycling and implications for climate on ancient Earth. *Journal of Geophysical Research-Planets* **106**, 1373–1399 (2001).

[8] Richter, F. M. Kelvin and the age of the Earth. *Journal of Geology* **94**, 395 (1985).