STOCHASTIC SIMULATION OPTIMIZATION OF WATER RESOURCES & MANAGEMENT

FINAL EXAMINATION

TIME ALLOCATION: 3 HOURS

INSTRUCTION TO STUDENTS: ATTEMPT **ALL** QUESTIONS

QUESTION 1 AND ANSWER (20 Marks)

A reservoir is planned both for gravity and lift irrigation through withdrawals from its storage. The total storage available for both the uses is limited to 5 units each year. It is decided to limit the gravity irrigation withdrawals in a year to 4 units. If X_1 is the allocation of water for gravity irrigation and X_2 is the allocation for lift irrigation, the two objectives planned to be maximized are expressed as

$$\text{Maximize} \quad Z_1(X) = 3x_1 - 2x_2 \quad \text{and} \quad Z_2(X) = -x_1 + 4x_2$$

For above problem, do the following

(i) Generate a Pareto Front of non-inferior (efficient) solutions by plotting Decision space and Objective space.

(ii) Formulate multi objective optimization model using weighting approach with w_1 and w_2 as weights for gravity and lift irrigation respectively.

(iii) Solve it, for (i) w_1=1 and w_2=2 (ii) w_1=2 and w_2=1

(iv) Formulate the problem using constraint method

Solution:

Formulation:

Objective functions:

Maximize $Z_1(X) = 3x_1 - 2x_2$ and $Z_2(X) = -x_1 + 4x_2$

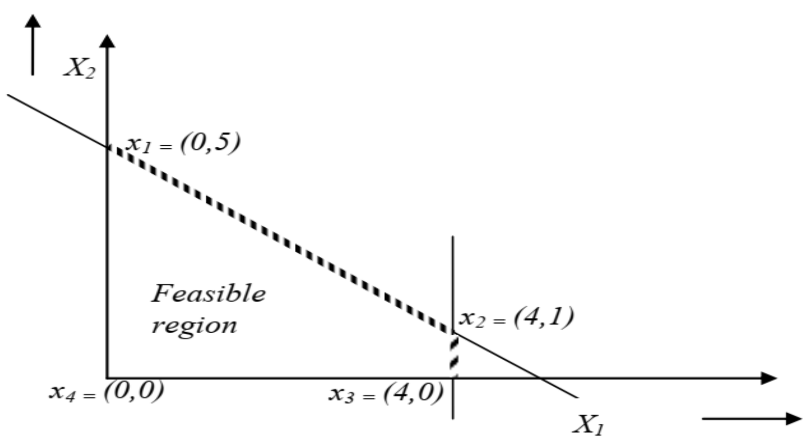
Subject to: $X_1 + X_2 \leq 5$;

$X_1 \leq 5$;

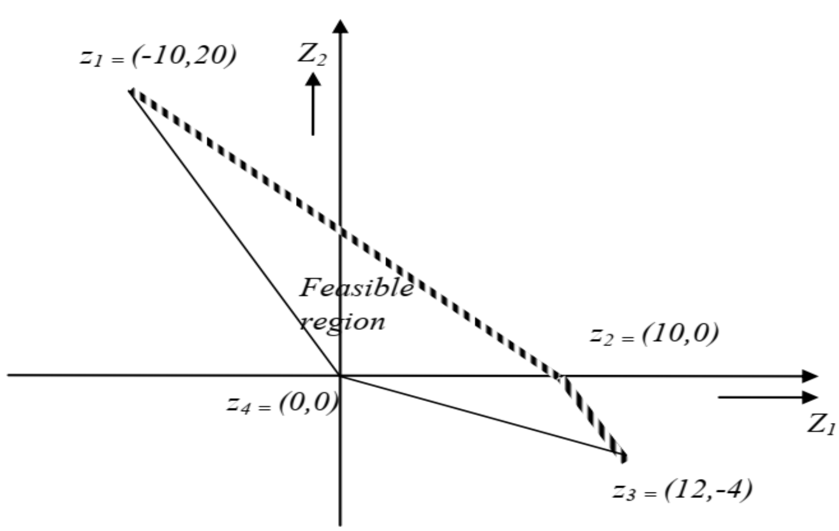
$X_1 \geq 0$;

$X_2 \geq 0$.

(i) Pareto front of non-inferior solutions in decision space



Pareto front of non-inferior solutions in objective space



(ii) Formulation of optimization problem using weighing method

Objective functions:

$$\text{Maximize} \quad Z = w_1 Z_1 + w_2 Z_2 = w_1 (3x_1 - 2x_2) + w_2 (- x_1 + 4x_2)$$

Subject to: $X_1 + X_2 \leq 5$; $X_1 \leq 5$; $X_1 \geq 0$; $X_2 \geq 0$

(iii) (a) w_1=1 and w_2=2

$$Z = (3x_1 - 2x_2) + 2 (- x_1 + 4x_2)$$

$$= x_1 + 6 x_2$$

The Z-line has a slope of -1/6 in decision space and Z has a maximum value of 30 at point (0,5).

(b) w_1=2 and w_2=1

$$Z = 2 (3x_1 - 2x_2) + (- x_1 + 4x_2)$$

$$= 5x_1$$

Z has a maximum value of 20 at points (4,1) and (4,0).

(iv) Formulation of optimization problem using constraint method

$$\text{Maximize} \quad Z_1 = 3x_1 - 2x_2$$

Subject to: $Z_2 = (- x_1 + 4x_2) \geq L_2$;

$X_1 + X_2 \leq 5$;

$X_1 \leq 5$;

$X_1 \geq 0$;

$X_2 \geq 0$.

QUESTION 2 AND ANSWER (20 Marks)

Probability density function (PDF) of a random variable X is

$$f(x) = 6\,x^2 \qquad 0 \le x \le 1$$

$$= 0 \qquad \text{else where}$$

Determine (1) Cumulative distribution function (cdf); (2) Expected value, E(X); (3) Variance, Var (X); (4) P[$X \ge 0.6$]; and (5) P[$0.4 \le X \le 0.7$]

Solution:

1. Cumulative distribution function

$$F(x) = \int_{-\infty}^{x} f(x)\,dx$$

$$= \int_{0}^{x} 6x^2\,dx = 2x^3 \qquad 0 \le x \le 1$$

2. Expected value, E(X)

$$E[X] = \int_{-\infty}^{\infty} x f(x)\,dx = \int_{0}^{1} x\,6x^2\,dx = 1.5$$

3. Variance, Var (X)

$$Var(X) = \int_{-\infty}^{\infty} (x-\mu)^2 f(x)\,dx$$

$$= \int_{0}^{1} (x-3/2)^2\,6x^2\,dx = 1.2$$

4. $P[X \geq 0.6]$

$$P[X \geq 0.6] = 1 - P[X \leq 0.6] = 1 - F(0.6)$$
$$= 1 - 2 \times 0.6^3 = 0.568$$

5. $P[0.4 \leq X \leq 0.7]$

$$P[0.4 \leq X \leq 0.7] = P[X \leq 0.7] - P[X \leq 0.4]$$
$$= F(0.7) - F(0.4)$$

QUESTION 3 AND ANSWER (20 Marks)

The monthly streamflow at a reservoir site is represented by a random variable X which follows normal distribution with a mean of 100 units and a standard deviation of 50 units. Find (1) $P[X > 150]$; (2) $P[X \leq 40]$ and (3) The flow value which will be exceeded with a probability of 0.8.

Solution:

The monthly streamflow at a reservoir site is represented by a random variable X which

(1) $P[X > 150]$

$$P[X > 150] = P[(X - \mu)/\sigma \geq (150 - 100)/50]$$
$$= P[Z \geq 1] = 1 - P[Z \leq 1]$$
$$= 1 - 0.8413 = 0.1587$$

(2) $P[X \leq 40]$

$$
\begin{aligned}
P[X \le 40] &= P[(X-\mu)/\sigma \le (40-100)/50] \\
&= P[Z \le -1.2] \\
&= 0.1539
\end{aligned}
$$

(3) To find $P[X \geq x] = 0.8$

$$
\begin{aligned}
&P[X \ge x] = 0.8 \\
&P[Z \ge (x-\mu)/\sigma] = 0.8 \\
&1 - P[Z \le z] = 0.8 \\
&P[Z \le z] = 0.2 \\
&z \;= (x-100)/50 = -0.84 \\
&x = 58 \ units
\end{aligned}
$$

QUESTION 4 AND ANSWER (20 Marks)

The time to failure of a pump in a water distribution system is assumed to follow an exponential distribution with the parameter $\lambda = 0.0137$/day (5 failures per year). Compute the reliability for 10 days operation.

Solution

The failure density function is an exponential distribution function

$$f_T(t) = \lambda\, e^{-\lambda t} = 0.0137\, e^{-0.0137\, t}, \; t \geq 0$$

The reliability is

$$
\begin{aligned}
\alpha(t) &= \int_t^\infty e^{-\lambda t}\, dt \\
&= e^{-\lambda t} = e^{-0.0137 t}, \qquad t \ge 0
\end{aligned}
$$

For 10 days, $\alpha = e^{-0.0137 * 10} = 0.872$

QUESTION 5 AND ANSWER (20 Marks)

A farming company wanted to decide on the size and number of pumps required for lift irrigation. Four differently sized pumps (x_1 through x_4) were considered. The objective was to minimize cost and the constraints were to supply water to all fields (who have a strong seasonally fluctuating demand). That meant certain quantities had to be supplied (quantity constraint) and a minimum number of fields per day had to be supplied (routing constraint). For other reasons, it was required that at least 6 of the smallest pumps should be included. The management wanted to use quantitative analysis and agreed to the following suggested linear programming approach. The available budget is Rs. 42 lakhs.

The optimization problem is

Minimize $41{,}400\, x_1 + 44{,}300\, x_2 + 48{,}100\, x_3 + 49{,}100\, x_4$

Subject to

$$0.84\, x_1 + 1.44\, x_2 + 2.16\, x_3 + 2.4\, x_4 \geq 170$$

$$16x_1 + 16\, x_2 + 16\, x_3 + 16\, x_4 \geq 1300$$

$$x_1 \geq 6$$

$$x_2, x_3, x_4 \geq 0$$

The solution of this problem using classical LP is

Min Cost = Rs. 38,64,975

$x_1 = 6,\ x_2 = 16.29,\ x_3 = 0,\ x_4 = 58.96.$

Fuzzy LP

As the demand forecasts had been used to formulate the constraints, there was a danger of not being able to meet higher demands. It is safe to stay below the available budget of Rs. 42 lakhs. Therefore, bounds and spread of the tolerance interval are fixed as follows:

Bounds: $d_1 = 37{,}00{,}000;\ d_2 = 170;\ d_3 = 1{,}300;\ d_4 = 6$

Spreads: $p_1 = 5{,}00{,}000;\ p_2 = 10;\ p_3 = 100;\ p_4 = 6$

Objective function in the classical LP problem is transformed as a constraint

$$41{,}400\, x_1 + 44{,}300\, x_2 + 48{,}100\, x_3 + 49{,}100\, x_4 + \lambda \leq 42{,}00{,}000$$

The optimization problem constraints are (acc. to eqn. 13)

Maximize λ

Subject to

$$0.083\,x_1 + 0.089\,x_2 + 0.096\,x_3 + 0.098\,x_4 + \lambda \leq 8.4$$

$$0.084\,x_1 + 0.144\,x_2 + 0.216\,x_3 + 0.240\,x_4 - \lambda \geq 17$$

$$0.16\,x_1 + 0.16\,x_2 + 0.16\,x_3 + 0.16\,x_4 - \lambda \geq 13$$

$$0.167\,x_1 - \lambda \geq 1$$

$$\lambda, x_2, x_3, x_4 \geq 0$$

Solutions obtained using classical and fuzzy LP are given below:

Classical LP	**Fuzzy LP**
$Z = 38{,}64{,}975$	$Z = 39{,}88{,}250$
$x_1 = 6$; $x_2 = 16.29$; $x_4 = 59.96$	$x_1 = 17.41$; $x_2 = 0$; $x_4 = 66.54$
Constraints:	
1. 170	1. 174.33
2. 1300	2. 1343.328
3. 6	3. 17.414

Through Fuzzy LP, a "leeway" has been provided with respect to all constraints and at additional cost of 3.2%. The decision maker is not forced into a precise formulation because of mathematical reasons even though he/she might only be able or willing to describe his/her problem in fuzzy terms.