STOCHASTIC DP

INTRODUCTION

In the previous lecture we discussed about stochastic processes and transition probabilities. In this lecture we will discuss the application of Stochastic Dynamic Programming (SDP) in reservoir operation.

While applying SDP to the reservoir operation problem, inflow to the reservoir is considered as a random variable. The reservoir storage at the beginning of period t and release during the period t are treated as state variables. All variables involved in the decision process, such as the reservoir storage, inflow, and release are discretized into a finite number of class intervals. A class interval for a variable has a representative value, generally taken as its midpoint.

SDP - RESERVOIR OPERATION

Notations

Q denotes the inflow; i and j are the class intervals (also referred to as states) of inflow in period t and period $t+1$, respectively; The representative values of inflow for the class i in period t and class j in period t+1 are denoted by Q_{it} *and* $Q_{j,t+1}$, respectively.

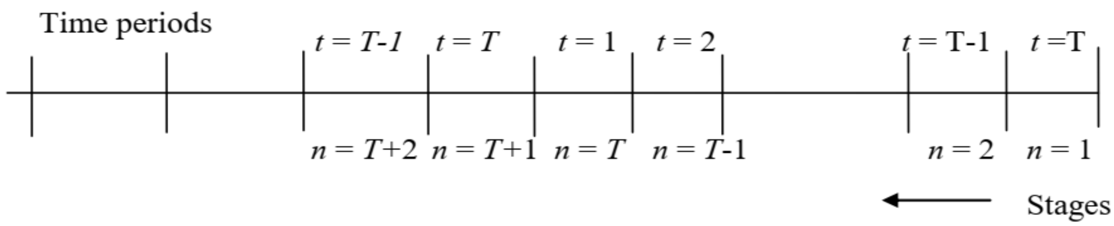
S denotes the reservoir storage; and k and l are the storage class intervals in periods t and t +1, respectively. The representative values for storage in the class intervals k and l are denoted by S_{kt}, and $S_{l,t+1}$, respectively.

Then according to the storage continuity, it can be expressed as

$$R_{kilt} = S_{kt} + Q_{it} - E_{klt} - S_{l,t+1}$$

where R_{kilt} is the reservoir release corresponding to the initial reservoir storage S_{kt}, the final reservoir storage $S_{l,t+1}$, and the evaporation loss E_{klt}. The loss E_{klt}, depends on the initial and final reservoir storages, S_{kt} and $S_{l,t+1}$. Since the inflow Q is a random variable, the reservoir storage and the release are also random variables.

The system performance measure depends on the state of the system defined by the storage class intervals k and l, and the inflow class interval i for the period t. We denote the system performance measure for a period t as B_{kilt} which corresponds to an initial storage state k, inflow state i, and final storage state l in period t. The system performance measure may be, for example, the amount of hydropower generated when a release of R_{kilt} is made from the reservoir, and the reservoir storages (which determine the head available for power generation) at the beginning and end of the period are respectively S_{kt} and $S_{l,t+1}$. Following backward recursion, the computations are assumed to start at the last period T of a distant year in the future and proceed backwards. Each time period denotes a stage in the dynamic programming. That is, $n = 1$ when $t=T$; $n=2$ when $t = T - 1$, etc. The index t takes values from T to 1, and the index n progressively increases with the stages in the SDP.



RECURSIVE RELATIONSHIP

Let $f_n^t(k, i)$ denote the maximum expected value of the system performance measure up to the end of the last period T (i.e. for periods t, t + 1, ..., T), when n stages are remaining, and the time period corresponds to t. With only one stage remaining (i.e. $n = 1$ and $t = T$),

$$f_1^T(k,i) = \underset{\{feasible\ l\}}{Max}\left[B_{kilT}\right] \qquad \forall k,i$$

Note that for a given k and i, only those values of l are feasible that result in a non-negative value of release, R_{kilt}. Since this is the last period in computation, the performance measure B_{kilt} is determined with certainty for the known values of k, i and l.

When we move to the next stage, $(n = 2,\ t = T - 1)$, the maximum value of the expected performance of the system is written as

$$f_2^{T-1}(k,i) = \underset{\{feasible\ l\}}{Max}\left[B_{kil\,T-1} + \sum_j p_{ij}^{T-1} f_1^T(l,j)\right] \qquad \forall k,i$$

When the computations are carried out for stage 2, period $T - 1$, the inflow during the period is known. However, since we are interested in obtaining the maximum expected system performance up to the end of the last period T, we must know the inflow during the succeeding period T also. Since this is not known with certainty, the expected value of the system performance is got by using the inflow transition probability P_{ij}^{T-1} for the period $T - 1$. It must be noted that the term within the summation denotes the maximized expected value of the system performance up to the end of the last period T, when the inflow state during the period $T - 1$ is i. The search for the optimum value of the performance is made over the end-of-the period storage l. Since $f_1^T(k, i)$ is already determined in stage 1, for all values of k and i, $f_2^{T-1}(k, i)$ given by above equation may be determined. The term {feasible l}, indicates that the search is made only over those end-of-the-period storages which result in a non-negative release R_{kilt} or satisfy any other constraints.

The relationship may be generalized for any stage n and period t as

$$F_n^t(k,i) = Max\left[B_{kil\,T-1} + \sum_j P_{ij}^t f_{n-1}^{t+1}(l,j)\right] \qquad \forall k,i$$
$$\{feasible\ l\}$$

Solving the equation recursively will yield a steady state policy within a few annual cycles, if the inflow transition probabilities P_{ij}^t are assumed to remain the same every year, which implies that the reservoir inflows constitute a stationary stochastic process. In general, the steady state is reached when the expected annual system performance, $[f^t_{n+T}(k, i) - f^t_n(k, i)]$ remains constant for all values of k, i, and t. When the steady state is reached, the optimal end-of-the-period storage class intervals, l, are defined for given k and i for every period t in the year. This defines the optimal steady state policy and is denoted by $l^*(k, i, t)$.

Example

Obtain steady state policy for the following data, when the objective is to minimize the expected value of the sum of the square of deviations of release and storage from their respective targets, over a year with two periods. Neglect evaporation loss. If the release is greater than the release target, the deviation is set to zero. Target Storage, T_S=30; Target Release, T_r=30; $B_{kilt} = (R_{kilt} - T_r)^2 + (S_k^t - T_s)^2$

Inflow transition probabilities

	t = 2				t = 1		
		j				j	
	i	1	2		i	1	2
t = 1	1	0.4	0.6	t = 2	1	0.5	0.5
	2	0.3	0.7		2	0.8	0.2

Inflow and Storage

For period 1

i	Q_{it}	k	S_{kt}
1	20	1	35
2	25	2	40

For period 2

i	Q_{it}	k	S_{kt}
1	30	1	25
2	45	2	30

Solution:

B_{kilt} values for all *k, i, l,* and *t*

For period 1

k	S_k^t	i	Q_k^t	l	S_k^{t+1}	E_{kilt}	R_{kilt}	$(S_k^t-T_s)^2$	$(R_{kilt}-T_r)^2$	B_{kilt}
1	35	1	20	1	25	0	30	25	0	25
1	35	1	20	2	30	0	25	25	25	50
1	35	2	25	1	25	0	35	25	0	25
1	35	2	25	2	30	0	30	25	0	25
2	40	1	20	1	25	0	35	100	0	100
2	40	1	20	2	30	0	30	100	0	100
2	40	2	25	1	25	0	40	100	0	100
2	40	2	25	2	30	0	35	100	0	100

For period 2

k	S_k^t	i	Q_k^t	l	S_k^{t+1}	E_{kilt}	R_{kilt}	$(S_k^t-T_s)^2$	$(R_{kilt}-T_r)^2$	B_{kilt}
1	25	1	30	1	35	0	20	25	100	0
1	25	1	30	2	40	0	15	25	225	250
1	25	2	45	1	35	0	35	25	0	25
1	25	2	45	2	40	0	30	25	0	25
2	30	1	30	1	35	0	25	0	25	25
2	30	1	30	2	40	0	20	0	100	100
2	30	2	45	1	35	0	40	0	0	0
2	30	2	45	2	40	0	35	0	0	0

<u>*n*=1, *t*=2</u>

$$f_1^2(k,i) = \underset{\text{feasible } l}{Min}\left[B_{kilT}\right] \qquad \forall k,i$$

$\underline{k}$	i	B_{kilt}		$f_1^2(k,i)$	l^*
		$l=1$	$l=2$		
1	1	125	250	125	1
1	2	25	25	25	1, 2
2	1	25	100	25	1
2	2	0	0	0	1, 2

<u>*n*=2, *t*=1</u>

$$f_2^1(k,i) = \underset{\text{feasible } l}{Min}\left[B_{kil1} + \sum_j p_{ij}^1 f_1^2(l,j)\right] \qquad \forall k,i$$

$$k=1; i=1; l=1;\ \ B_{kil1} + \sum_j p_{ij}^1 f_1^2(l,j) = 25 + 0.4*125 + 0.6*25 = 90$$

$$k=1; i=1; l=2;\ \ B_{kil1} + \sum_j p_{ij}^1 f_1^2(l,j) = 50 + 0.4*25 + 0.6*0 = 60$$

$$k=1; i=2; l=1;\ \ B_{kil1} + \sum_j p_{ij}^1 f_1^2(l,j) = 25 + 0.3*125 + 0.7*25 = 80$$

$$k=1; i=2; l=2;\ \ B_{kil1} + \sum_j p_{ij}^1 f_1^2(l,j) = 25 + 0.3*25 + 0.7*0 = 32.5$$

$$k=2; i=1; l=1;\ \ B_{kil1} + \sum_j p_{ij}^1 f_1^2(l,j) = 100 + 0.4*125 + 0.6*25 = 165$$

$\underline{k}$	i	B_{kilt}		$f_2^1(k,i)$	l^*
		$l=1$	$l=2$		
1	1	90	60	60	2
1	2	80	32.5	32.5	2
2	1	165	110	110	2
2	2	155	107.5	107.5	2

<u>*n*=3, *t*=2</u>

$$f_3^2(k,i) = \underset{\text{feasible } l}{Min}\left[B_{kil2} + \sum_j p_{ij}^1 f_2^1(l,j)\right] \qquad \forall k,i$$

$\underline{k}$	i	B_{kilt}		$f_3^2(k,i)$	l^*
		$l=1$	$l=2$		
1	1	171.25	358.75	171.25	1
1	2	79.5	220.7	79.5	1
2	1	71.25	208.75	71.25	1
2	2	54.5	109.5	54.5	1

n=4, t=1

$$f_4^1(k,i) = \underset{\text{feasible } l}{Min}\left[B_{kil1} + \sum_j p_{ij}^2 f_3^2(l,j)\right] \qquad \forall k,i$$

k	i	B_{kilt}		$f_4^1(k,i)$	l*
		l =1	l =2		
1	1	141.2	111.2	111.2	2
1	2	132.025	84.525	84.525	2
2	1	216.2	161.2	161.2	2
2	2	207.025	159.525	159.525	2

n=5, t=2

k	i	B_{kilt}		$f_5^2(k,i)$	l*
		l =1	l =2		
1	1	222.86	410.36	222.86	1
1	2	130.865	185.865	130.865	1
2	1	122.86	260.36	122.86	1
2	2	105.865	160.865	105.865	1

n=6, t=1

k	i	B_{kilt}		$f_6^1(k,i)$	l*
		l =1	l =2		
1	1	192.664	162.66	162.66	2
1	2	183.46	135.96	135.96	2
2	1	267.66	212.66	212.66	2
2	2	258.46	210.96	210.96	2

n=7, t=2

k	i	B_{kilt}		$f_7^2(k,i)$	l*
		l =1	l =2		
1	1	274.31	461.81	274.31	1
1	2	182.32	237.32	182.32	1
2	1	174.31	311.81	174.31	1
2	2	157.32	212.32	157.32	1

n=8, t=1

k	i	B_{kilt}		$f_8^1\ (k,i)$	l^*
		l =1	l =2		
1	1	244.12	214.12	214.12	2
1	2	234.92	187.42	187.42	2
2	1	319.12	264.12	264.12	2
2	2	309.92	262.42	262.42	2

The computations are terminated after this stage because it is verified that the annual system performance measure remains constant. $f_8^1\ (1,1) - f_6^1\ (1,1) = 214.12 - 162.66 = 51.46$.

Steady state policy for period 1

k	i	l^*
1	1	2
1	2	2
2	1	2
2	2	2

Steady state policy for period 2

k	i	l^*
1	1	1
1	2	1
2	1	1
2	2	1

BIBLIOGRAPHY / FURTHER READING:

1. Jain, S.K. and V.P. Singh, Water Resources Systems Planning and Management, Vol. 51, Elsevier Science, 2003.
2. Loucks D.P. and van Beek E., 'Water Resources Systems Planning and Management', UNESCO Publishing, The Netherlands, 2005.
3. Loucks, D.P., J.R. Stedinger, and D.A. Haith, *Water Resources Systems Planning and Analysis*, Prentice-Hall, N.J., 1981.
4. Rao S.S., *Engineering Optimization – Theory and Practice*, Fourth Edition, John Wiley and Sons, 2009.
5. Vedula S., and P.P. Mujumdar, *Water Resources Systems: Modelling Techniques and Analysis*, Tata McGraw Hill, New Delhi, 2005.