

**CHANCE CONSTRAINED LP - II****INTRODUCTION**

In the previous lecture, we explained the basic concepts of Chance Constrained Linear Programming and the various types. In this lecture we will address the application of CCLP in reservoir operation.

In reservoir planning and operation problems, the future inflow,  $Q_t$ , is a random variable and is not known with certainty. Its probability distribution, however, may be estimated from the historical sequence of inflows. The storage  $S_t$  and release  $R_t$ , being functions of inflow,  $Q_t$ , are also random variables. In a constraint containing two random variables, if the probability distribution of one is known, the probabilistic behavior of the second can be expressed as a measure of chance in terms of the probability of the first variable. If a constraint contains more than two random variables, we get into computational complications, and we need to understand the specific problem clearly to reformulate the problem, if necessary, and avoid those complications.

**CHANCE CONSTRAINT**

The constraint, relating the release,  $R_t$ , (random) and demand,  $D_t$ , (deterministic), is expressed as a chance constraint,  $P [R_t \geq D_t] \geq \alpha_1$ ; It means that the probability of release equaling or exceeding the known demand is at least equal to  $\alpha_1$ , which is referred to as the reliability level. The interpretation of this chance constraint is simply that the reliability of meeting the demand in period  $t$  is at least  $\alpha_1$ . Similarly, chance constraints for the maximum release and the maximum and minimum storage can be written as

$$P [R_t \leq R_t^{max}] \geq \alpha_2$$

$$P [S_t \leq K] \geq \alpha_3$$

$$P [S_t \geq S_{min}] \geq \alpha_4$$

To use the above chance constraints in an optimization algorithm, we must first determine the probability distribution of  $R_t$  and  $S_t$  from the known probability distribution of  $Q_t$ . However, because  $S_t$ ,  $Q_t$  and  $R_t$  are all interdependent through the continuity equation, it is, in general, not possible to derive the probability distributions of both  $S_t$  and  $R_t$  simultaneously. To

overcome this difficulty and to enable the use of linear programming in the solution, a linear decision rule is appropriately defined.

### **LINEAR DECISION RULE**

The linear decision rule (LDR) relates the release,  $R_t$ , from the reservoir as a linear function of the water available in period  $t$ . The simplest form of such an LDR is

$$R_t = S_t + Q_t - b_t$$

where  $b_t$  is a deterministic parameter called the decision parameter.

In this LDR, the entire amount,  $Q_t$ , is taken into account while making the release decision. Depending on the proportion of inflow,  $Q_t$ , used in the linear decision rule, a number of such LDRs may be formulated. A general form of this LDR may be written as

$$R_t = S_t + \beta_t Q_t - b_t \quad 0 \leq \beta_t \leq 1$$

$\beta_t = 0$  yields a relatively conservative release policy with release decisions related only to the storage,  $S_t$ ;  $\beta_t = 1$  yields an optimistic policy where the entire amount of water available ( $S_t + Q_t$ ), is used in the LDR. Consider the LDR

$$R_t = S_t + Q_t - b_t$$

Storage continuity equation is

$$S_{t+1} = S_t + Q_t - R_t$$

Using above two equations

$$S_{t+1} = b_t$$

Thus, the random variable,  $S_{t+1}$ , is set equal to a deterministic parameter  $b_t$ . Thus, the role of the LDR in this case is to treat  $S_t$ , as deterministic in formulation. A main advantage of doing this is to do away with one of the random variables,  $S_t$ , so that the distribution of the other random variable,  $R_t$ , may be expressed in terms of the known distribution of  $Q_t$ . This implies that the variance of  $Q_t$  is entirely transferred to the variance of  $R_t$ .

Assuming that the evaporation loss  $E_t$  is a linear function of the average storage volume,  $e_{0t} + e_t (S_t + S_{t+1})/2$ , where  $e_{0t}$  is the fixed evaporation volume loss and  $e_t$  is the evaporation volume loss per unit average storage volume in period  $t$ , then the linear decision rule can be written

$$\text{as, } R_t = Q_t - e_{0t} + \left(1 - \frac{e_t}{2}\right)b_{t-1} - \left(1 + \frac{e_t}{2}\right)b_t.$$

**Deterministic Equivalent of Chance Constraint**

Knowing the probability distribution of inflow,  $Q_t$ , it is possible to obtain the deterministic equivalents of the chance constraints using the LDR, as follows:

$$P [R_t \geq D_t] \geq \alpha_1$$

$$P [S_t + Q_t - b_t \geq D_t] \geq \alpha_1$$

$$P [b_{t-1} + Q_t - b_t \geq D_t] \geq \alpha_1$$

$$P [Q_t \geq D_t + b_t - b_{t-1}] \geq \alpha_1$$

$$P [Q_t \leq D_t + b_t - b_{t-1}] \leq 1 - \alpha_1$$

The term  $D_t + b_t - b_{t-1}$ , is deterministic with  $b_t$  and  $b_{t-1}$  being decision variables and  $D_t$  being a known quantity for the period  $t$ .

$$F_{Q_t}(D_t + b_t - b_{t-1}) \leq 1 - \alpha_1$$

$$(D_t + b_t - b_{t-1}) \leq F_{Q_t}^{-1}(1 - \alpha_1)$$

$F_{Q_t}^{-1}(1 - \alpha_1)$  is the flow,  $q_t$ , at which the CDF value is  $(1 - \alpha_1)$ . Similarly, deterministic equivalent of chance constraint

$$P(R_t \leq R_t^{max}) \geq \alpha_2 \text{ is}$$

$$R_t^{max} + b_t - b_{t-1} \geq F_{Q_t}^{-1}(\alpha_2)$$

Since the storage,  $S_{t+1}$ , is set equal to the deterministic parameter,  $b_t$ , the chance constraints containing only the storage random variable are written as deterministic constraints (without using the probability distribution of inflows).

**CHANCE CONSTRAINED LP**

Complete deterministic equivalent of the CCLP is thus written as

$$\text{Min } K$$

Subject to

$$(D_t + b_t - b_{t-1}) \leq F_{Q_t}^{-1}(1 - \alpha_1) \quad \text{for all } t$$

$$R_t^{max} + b_t - b_{t-1} \geq F_{Q_t}^{-1}(\alpha_2) \quad \text{for all } t$$

$$b_{t-1} \leq K \quad \text{for all } t$$

$$b_{t-1} \geq S_{min} \quad \text{for all } t$$

$$b_t \geq 0 \quad \text{for all } t$$

$$K \geq 0 \quad \text{for all } t$$

While solving this model, for a problem with 12 periods (months) in a year, we also set  $b_0 = b_{12}$  for a steady state solution. Further, depending on the nature of LDR used, the decision

parameters,  $b_t$ , may be unrestricted in sign. For example, if we use the LDR,  $R_t = S_t - b_t$ , the decision parameter,  $b_t$ , may be allowed to take negative values. CCLP can also be applied for reliability based reservoir sizing.

**Example Problem:**

For the following chance constrained optimization problem, formulate the equivalent deterministic optimization problem using the LDR,  $R_t = S_t - b_t$ . Storage continuity should be maintained. Neglect losses. Following table gives the  $F^{-1}()$  values for the inflows and  $R_{max}$  and  $R_{min}$  values for different periods.

Minimize  $K$

$$P [S_{min} \leq S_t \leq K] \geq 0.8 \quad \forall t$$

$$P [R_t \leq R_t^{max}] \geq 0.85 \quad \forall t$$

$$P [R_t \geq D_t] \geq 0.75 \quad \forall t$$

t	$F^{-1}(0.0)$	$F^{-1}(0.2)$	$F^{-1}(0.25)$	$F^{-1}(0.75)$	$F^{-1}(0.8)$	$F^{-1}(0.85)$	$R_{max}$	$R_{min}$	$S_{min}$
1	0	12	33	60	90	93	90	24	2
2	0	3	20	48	60	80	84	20	2
3	0	6	21	36	72	85	84	20	2

**Solution**

Deterministic equivalent of  $P [S_{min} \leq S_t \leq K] \geq 0.8 \quad \forall t$ .

This can be divided into two constraints as  $P [S_{min} \leq S_t] \geq 0.8$  and  $P [S_t \leq K] \geq 0.8$ .

Linear decision rule is given as  $R_t = S_t - b_t$ .

Then, continuity equation can be written as

$$\begin{aligned} S_{t+1} &= S_t + Q_t - R \\ &= S_t + Q_t - S_t + b_t \\ &= Q_t + b_t \end{aligned}$$

Hence,  $S_t = Q_{t-1} + b_{t-1}$

Deterministic equivalent of  $P [S_{min} \leq S_t] \geq 0.8$

$$P [S_{min} \leq Q_{t-1} + b_{t-1}] \geq 0.8$$

$$P [Q_{t-1} \geq S_{min} - b_{t-1}] \geq 0.8$$

$$P [Q_{t-1} \leq S_{min} - b_{t-1}] \leq (1 - 0.8)$$

$$P [Q_{t-1} \leq 2 - b_{t-1}] \leq 0.2$$

$$F_{Q_{t-1}}(2 - b_{t-1}) \leq 0.2$$

$$(2 - b_{t-1}) \leq F_{Q_{t-1}}^{-1}(0.2)$$

Therefore,

$$(2 - b_3) \leq 6 \quad t = 1$$

$$(2 - b_1) \leq 12 \quad t = 2$$

$$(2 - b_2) \leq 3 \quad t = 3$$

Deterministic equivalent of  $P [S_t \leq K] \geq 0.8$ .

$$P [S_t \leq K] \geq 0.8$$

$$P [Q_{t-1} + b_{t-1} \leq K] \geq 0.8$$

$$P [Q_{t-1} \leq K - b_{t-1}] \geq 0.8$$

$$K - b_{t-1} \geq F_{Q_{t-1}}^{-1}(0.8)$$

Therefore,

$$K - b_3 \geq 72 \quad t = 1$$

$$K - b_1 \geq 90 \quad t = 2$$

$$K - b_2 \geq 60 \quad t = 3$$

Deterministic equivalent of  $P [R_t \leq R_t^{max}] \geq 0.85$

$$P [S_t - b_t \leq R_t^{max}] \geq 0.85$$

$$P [Q_{t-1} + b_{t-1} - b_t \leq R_t^{max}] \geq 0.85$$

$$P [Q_{t-1} \leq R_t^{max} - b_{t-1} + b_t] \geq 0.85$$

$$R_t^{max} - b_{t-1} + b_t \geq F_{Q_{t-1}}^{-1}(0.85)$$

$$90 - b_3 + b_1 \geq 85 \quad t = 1$$

$$84 - b_1 + b_2 \geq 93 \quad t = 2$$

$$84 - b_2 + b_3 \geq 80 \quad t = 3$$

Deterministic equivalent of  $P [R_t \geq D_t] \geq 0.75$

$$P [S_t - b_t \geq D_t] \geq 0.75$$

$$P [Q_{t-1} + b_{t-1} - b_t \geq D_t] \geq 0.75$$

$$P [Q_{t-1} \geq D_t - b_{t-1} + b_t] \geq 0.75$$

$$D_t - b_{t-1} + b_t \leq F_{Q_{t-1}}^{-1}(0.25)$$

$$24 - b_3 + b_1 \leq 21 \quad t = 1$$

$$20 - b_1 + b_2 \leq 33 \quad t = 2$$

$$20 - b_2 + b_3 \leq 20 \quad t = 3$$