CHANCE CONSTRAINED LP - I

INTRODUCTION

As discussed in lecture 1 of this module, the two commonly used ESO techniques are: Chance Constrained Linear Programming (CCLP), and Stochastic Dynamic Programming (SDP). In this lecture we will discuss the Chance Constrained Linear Programming and its formulation.

Chance Constraint

In system modeling, all the required quantities cannot be assessed with certainty. Such parameters in optimization model are therefore treated as random variables. Hence, while solving this optimization model using Linear Programming (LP) approach, some of the constraint coefficients (in RHS or LHS) may be uncertain. This questions the compliance of the particular constraint. Hence, the original deterministic constraint is replaced with a probabilistic statement in the form of a **chance-constraint** as

$$P\left\{\sum_{j=1}^{n} a_{ij} x_j \leq b_i\right\} \geq \alpha_i, \qquad i = 1,2,\ldots,m \tag{1}$$

where α_i is the specified reliability of compliance of the i^{th} constraint. For mathematical operations, the above probabilistic statement should be converted into a deterministic equivalent. The randomness can occur in the above equation in three ways: (i) only the RHS coefficients b_i are random; (ii) only the LHS elements a_{ij} are random and (iii) both a_{ij} and b_i are random.

(1) Random right hand side coefficients

This is the simplest case among the three mentioned above. The coefficient b_i will be expressed as B_i since it is random. Hence, eqn (1) can be expressed as,

$$P\left\{\sum_{j=1}^{n} a_{ij} x_j \leq B_i\right\} \geq \alpha_i, \qquad i = 1,2,\ldots,m \tag{2}$$

It can also be expressed as

$$P\left\{B_i \le \sum_{j=1}^{n} a_{ij} x_j\right\} \le 1-\alpha_i \tag{3}$$

Let the CDF of the random component B_i be $F_{Bi}(b)$ with mean μ_{Bi} and standard deviation σ_{Bi}. Then, eqn. (3) becomes

$$F_{B_i}\left(\sum_{j=1}^{n} a_{ij} x_j\right) \le 1-\alpha_i \tag{4}$$

Converting this into a standard normal variate $Z_{Bi} = (B_i - \mu_{Bi}) / \sigma_{Bi}$ (as explained in lecture 1 of this module), eqn. (4) can be written as

$$F_{Z_{B_i}}\left(\frac{\sum_{j=1}^{n} a_{ij} x_j - \mu_{B_i}}{\sigma_{B_i}}\right) \le 1-\alpha_i \tag{5}$$

Hence, the deterministic equivalent of the chance-constraint in eqn. (2) is

$$\left(\frac{\sum_{j=1}^{n} a_{ij} x_j - \mu_{B_i}}{\sigma_{B_i}}\right) \le F_{Z_{B_i}}^{-1}(1-\alpha_i) \tag{6}$$

Example:

Find the deterministic equivalent of the statement $P[5x_1 + 3x_2 \le K] \ge 0.8$. The coefficient K is uncertain and follows a normal distribution with a mean of 5 units and standard deviation of 1 unit.

Solution:

$P[5x_1 + 3x_2 \le K] \ge 0.8$

Deterministic equivalent is

$(5x_1 + 3x_2 - \mu_K) / \sigma_K \le F_K^{-1}(1 - 0.8)$

$(5x_1 + 3x_2 - 5) / 1 \le -0.84$

Therefore, the resulting deterministic equivalent is

$(5x_1 + 3x_2) \le 4.16$

<u>(2) Random technological coefficients a_{ij}</u>

The coefficient $\underline{a_{ij}}$ will be expressed as A_{ij} since it is random. The CDF of the random component A_{ij} is $F_{Aij}(a)$ with mean μ_{Aij} and standard deviation σ_{Aij}. Hence, eqn (1) can be expressed as,

$$P\left\{\sum_{j=1}^{n} A_{ij} x_j \le b_i\right\} \ge \alpha_i, \qquad i = 1,2,\ldots,m \tag{7}$$

It can also be expressed as

$$P(A_i \le b_i) \ge \alpha_i \tag{8}$$

where $A_i = \sum_{j=1}^{n} A_{ij} x_j$. Mean and variance of A_i can be derived as

$$\mu_{A_i} = \sum_{j=1}^{n} \mu_{A_{ij}} x_j \quad \text{and} \quad \sigma_{A_i}^2 = \sum_{j=1}^{n} \sigma_{A_i}^2 x_j^2 + 2\sum_{j=1}^{n-1} \sum_{j'=j+1}^{n} x_j x_{j'} Cov(A_{ij}, A_{ij'})$$

Hence,

$$P\left[Z_{A_i} = \frac{A_i - \mu_{A_i}}{\sigma_{A_i}} \le \frac{b_i - \mu_{A_i}}{\sigma_{A_i}}\right] \ge \alpha_i \tag{9}$$

If all A_{ij}s follow a normal distribution, then

$$F_{A_i}\left(\frac{b_i - \mu_{A_i}}{\sigma_{A_i}}\right) \ge \alpha_i$$

or

$$\frac{b_i - \mu_{A_i}}{\sigma_{A_i}} \ge F_{A_i}^{-1}(\alpha_i) \tag{10}$$

<u>**Example:**</u>

Find the deterministic equivalent of the statement $P[A_{11} x_1 + x_2 \le 5] \ge 0.8$. The coefficient A_{11} is uncertain and follows a normal distribution with a mean of 3 units and standard deviation of 0.5 units. The variable x_2 is certain.

Solution:

$P[A_{11}\,x_1 + x_2 \le 5] \ge 0.8$

$A_1 = A_{11}\,x_1 + x_2$

Therefore, $\mu_{A1} = 3\,x_1 + x_2$ and $\sigma^2_{A1} = (0.5\,x_1)^2$

Deterministic equivalent is

$$\frac{5 - 3x_1 - x_2}{0.5x_1} \ge F_{A_i}^{-1}(0.8)$$

$$5 - 3x_1 - x_2 \ge 0.84 * 0.5x_1$$

Therefore, the resulting deterministic equivalent is

$$3.42x_1 + x_2 \le 5$$

(3) Both a_{ij} and b_j are random

Eqn (1) can be written as

$$P\left\{\sum_{j=1}^{n} A_{ij}x_j \le B_i\right\} \ge \alpha_i, \qquad i = 1,2,\ldots,m \tag{11}$$

Expressing $T_i = \sum_{j=1}^{n} A_{ij}x_j \le B_i$, similar to eqn. 10,

$$\frac{\mu_{B_i} - \mu_{A_i}}{\sigma_{T_i}} \ge F_{T_i}^{-1}(\alpha_i) \tag{12}$$

where $\mu_{A_i} = \sum_{j=1}^{n} \mu_{A_{ij}} x_j$

and $\sigma^2_{T_i} = \sum_{j=1}^{n} \sigma^2_{A_i} x_j^2 + \sigma^2_{B_i} + 2\sum_{j=1}^{n-1} \sum_{j'=j+1}^{n} x_j x_{j'} Cov(A_{ij}, A_{ij'}) + 2\sum_{j=1}^{n} x_j Cov(A_{ij}, B_i)$