UNCERTAINTY AND RELIABILITY ANALYSIS

INTRODUCTION

In the previous lecture the concept of probability, random variables and various probability distributions were dealt. In this lecture the major element of planning process i.e., uncertainty and also reliability analysis will be discussed.

UNCERTAINTY

Most water resources decision problems face the risk of uncertainty mainly because of the randomness of the variables that influence the performance of the systems. Uncertainty in water resources arises mainly due to the stochastic nature of hydrological processes such as rainfall, evaporation, temperature and also other variables like future population growth, per capita water usages, irrigation patterns etc. Normally, depending upon the severity of uncertainty of the quantities involved, they are replaced by either their expected value or some critical value (e.g., worst case value) and proceed with a deterministic approach. The expected value or median value is used when the uncertainty is reasonably small and the performance of the system is not much affected.

Sensitivity analysis

This is the simplest method for assessing the effect of uncertainty. The system performance is analysed by varying the magnitude of the more uncertain parameters. This will help to identify the most sensitive parameter or variable in a system. For example let the cost required for a structure of capacity k is $C(k) = ak^b$. The parameter b represents the elasticity of costs and $b = \frac{k}{C}\frac{dC}{dk}$ or $\frac{dC}{C} = b\frac{dk}{k}$. For a value of b = 0.6, a change in capacity by 10% will result in a cost change of only 6%.

First-order analysis or delta method

This is used to estimate the uncertainty in a deterministic model formulation in which the parameters involved are uncertain. For example, in the estimation of weir discharge $Q = CLH^{1.5}$, if the parameters C and H are not certain, then Q is also uncertain. Through first order analysis, one would be able to estimate the mean and variance of a random variable which is related to other variables which may also be random. The combined effect of uncertainty can thus be assessed. To elaborate the concept, consider a random variable X which is a function of n other random variables Mathematically, $X = f(\mathbf{Y})$, where $\mathbf{Y} = (Y_1, Y_2, \ldots, Y_n)$ is a vector of n variables. Through Taylor's expansion about the means of n random variable, the first order approximation of the r.v X can be expressed as (ignoring the second and higher order terms)

$$X = f(\bar{y}) + \sum_{i=1}^{n} \left[\frac{\partial f}{\partial Y_i}\right]_{Y=\bar{y}} (Y_i - \bar{y}_i) \tag{1}$$

where $\bar{y} = (\bar{y}_1, \bar{y}_2, \ldots, \bar{y}_n)$ and $\left[\frac{\partial f}{\partial Y_i}\right]_{\bar{\mathbf{y}}}$ is called the sensitivity coefficient which is the rate of change of function value $f(\mathbf{y})$ at $Y = \bar{y}$.

The mean and variance of the random variable X are approximated as

$$\mu_X = E(X) \approx f(\bar{y}) \tag{2}$$

$$Var(X) = Var(f(\bar{y})) + Var\left[\sum_{i=1}^{n} \left[\frac{\partial f}{\partial Y_i}\right]_{\bar{\mathbf{y}}} (Y_i - \bar{y}_i)\right] \tag{3}$$

$Var(f(\bar{y})) = 0$ since $f(\bar{y})$ is calculated using the mean values of $\bar{\mathbf{y}}$ and is constant.

Therefore,

$$Var(X) = Var\left[\sum_{i=1}^{n} a_i (Y_i - \bar{y}_i)\right] \tag{4}$$

where $a_i = \left[\frac{\partial f}{\partial Y_i}\right]_{\bar{\mathbf{y}}}$

Eqn. (4) can also be expressed as

$$Var(X) \approx \sum_{i=1}^{n} a_i^2 \sigma_i^2 + 2\sum_{i=1}^{n-1} \sum_{j=i+1}^{n} a_i a_j Cov(Y_i, Y_j) \tag{5}$$

where σ_i^2 is the variance of the r.v Y_i. For uncorrelated Y_i s $Cov(Y_i, Y_j) = 0$. Therefore,

$$Var(X) \approx \sum_{i=1}^{n} a_i^2 \sigma_i^2 \tag{6}$$

The coefficient of variation can be expressed as

$$(C_{v,X})^2 = \sum_{i=1}^{n} a_i^2 \left(\frac{\bar{y}_i}{\mu_Y}\right)(C_{v,Y_i})^2 \tag{7}$$

The relative contribution of uncertainty of each component $a_i^2\sigma_i^2$ to the total uncertainty as expressed in the above equations (6) or (7) can be used to minimize the effects of uncertainties in model output *X*.

Example:

Analyse the uncertainty in the channel discharge $Q = \frac{1}{n} AR^{2/3}S^{1/2}$ where the parameter *R* is certain and the parameters *n* and *S* are uncertain.

Solution:

Since *R* and *A* are certain $Q = C\,n^{-1}S^{1/2}$ where the constant $C = AR^{2/3}$.

First order approximation of Q is

$$Q \approx \overline{Q} + \left[\frac{\partial Q}{\partial n}\right]_{(\bar{n},\bar{S})}(n-\bar{n}) + \left[\frac{\partial Q}{\partial S}\right]_{(\bar{n},\bar{S})}(S-\bar{S})$$

$$= \overline{Q} + \left[-C\frac{S^{1/2}}{\bar{n}^2}\right](n-\bar{n}) + \left[C\frac{1}{2\bar{n}S^{-1/2}}\right]_{(\bar{n},\bar{S})}(S-\bar{S})$$

where $\overline{Q} = C\frac{1}{\bar{n}}\bar{S}^{1/2}$; $\frac{\partial Q}{\partial n}$ and $\frac{\partial Q}{\partial S}$ are sensitivity coefficients.

The uncertainty of *Q* assuming *n* and *S* as independent variables, can be expressed by the variance operator as

$$\sigma_Q^2 = \left[\frac{\partial Q}{\partial n}\right]^2_{(\bar{n},\bar{S})}\sigma_n^2 + \left[\frac{\partial Q}{\partial S}\right]^2_{(\bar{n},\bar{S})}\sigma_S^2$$

Expressing uncertainty in the form of coefficient of variance gives (as per eqn. 7)

$$(C_{v_Q})^2 = \left[\frac{\partial Q}{\partial n}\right]^2\left(\frac{\bar{n}}{\overline{Q}}\right)(C_{v_n})^2 + \left[\frac{\partial Q}{\partial S}\right]^2\left(\frac{\bar{S}}{\overline{Q}}\right)(C_{v_S})^2$$

$$= (C_{v_n})^2 + 0.25\,(C_{v_S})^2$$

<u>**RELIABILITY**</u>

The hydrosystems designed are expected to exhibit resistance against any external stresses. The resistance or strength of any system is its ability to function without failure against the external stresses. Reliability is the probability of the system functioning satisfactorily. Consider the state of a system denoted by a r.v X_t at time *t* for $t = 1,2,\ldots,n$. If the possible outcomes of X_t can be divided into two sets: (i) satisfactory outputs or successes, S and (ii) unsatisfactory outputs or failures, F. Then, reliability α can be expressed as

$$\alpha = P(X_t \in S)$$

For example, the reliability of a water supply system depends on the conditions when supply is greater than demand i.e., successes. Hence, reliability is the ratio of non-failures to the total periods.

Reliability Computation using load-resistance method

Reliability is expressed as the probability that the capacity of the system (or resistance) exceeds or equals the loading i.e., $\alpha = P(L \leq R)$. Risk is just the opposite of reliability or it is the probability of the loading exceeding the resistance i.e., $\alpha' = P(L > R) = 1 - \alpha$.

Computation by direct integration:

Let the joint PDF of load and resistance be $f_{L,R}(l, r)$. Then reliability is

$$\alpha = \int_0^\infty \int_0^r f_{L,R}(l, r)\, dl\, dr$$

If loading and resistance are independent, then,

$$\alpha = \int_0^\infty f_R(r) \left[\int_0^r f_L(l)\, dl \right] dr$$

$$= \int_0^\infty f_R(r) F_L(r)\, dr$$

where $F_L(r)$ is the CDF of the loading at $L = r$.

Computation using safety margin and safety factor

The safety margin (SM) can be defined as the difference between the system's resistance and the anticipated loading i.e., $SM = R - L$. Then reliability can be expressed as,

$$\alpha = P(R - L \geq 0) = P(SM \geq 0)$$

This requires the PDF of SM. Assuming that the loading and resistance are independent normal random variables, the mean and variance of SM are

$$\mu_{SM} = \mu_R - \mu_L \qquad \text{and} \qquad \sigma_{SM}^2 = \sigma_R^2 + \sigma_L^2$$

Then reliability is

$$\alpha = P\left(\frac{SM - \mu_{SM}}{\sigma_{SM}} \geq \frac{-\mu_{SM}}{\sigma_{SM}}\right)$$
$$= P\left(Z \leq \frac{\mu_{SM}}{\sigma_{SM}}\right) = \Phi\left(\frac{\mu_{SM}}{\sigma_{SM}}\right)$$

Example:

The average surface runoff to a sewer is 3 m^3/s with a standard deviation of 1.2 m^3/s. The mean capacity of the sewer is estimated to be 4.5 m^3/s with a standard deviation of 0.8 m^3/s. Compute the reliability using safety margin approach.

Solution:

$\mu_L = 3$; $\mu_R = 4.5$; $\sigma_L = 1.2$; $\sigma_R = 0.8$

$\mu_{SM} = \mu_R - \mu_L = 1.5$

$\sigma_{SM}^2 = \sigma_R^2 + \sigma_L^2 = 2.08$

Therefore, $\alpha = \left(Z \leq \frac{1.5}{\sqrt{2.08}}\right) = 0.851$.

And risk is, α' = 1 – 0.851 = 0.149

Reliability Computation using time-to-failure analysis

In this case, instead of considering the resistance and loading, only one r.v i.e., time is considered. The time-to-failure (T) of a system is the random variable with PDF $f_T(t)$ and is called the **failure density function**. Then, the reliability within a time interval [0, t] can be expressed as

$$\alpha(t) = \int_t^{\infty} f_T(t)\, dt$$

And risk can be expressed as $$\alpha'(t) = \int_0^{t} f_T(t)\, dt$$

Example

The time to failure of a pump in a water distribution system is assumed to follow an exponential distribution with the parameter $\lambda = 0.0137$/day (5 failures per year). Compute the reliability for 10 days operation.

Solution

The failure density function is an exponential distribution function

$$f_T(t) = \lambda\, e^{-\lambda t} = 0.0137\, e^{-0.0137\, t}\ ,\ t \geq 0$$

The reliability is

$$\alpha(t) = \int_t^\infty e^{-\lambda t}\ dt$$

$$= e^{-\lambda t} = e^{-0.0137t}, \qquad t \geq 0$$

For 10 days, $\alpha = e^{-0.0137 * 10} = 0.872$