WEIGHTING FUNCTION AND CONSTRAINT METHODS

INTRODUCTION

A multi-objective optimization problem essentially consists of two steps: plan formulation and plan selection. In the previous lectures, we have discussed about plan formulation and noninferior solutions. In the present discussion, we will introduce two techniques commonly adopted for plan selection: (i) Utility Function Method (Weighting function method) (ii) Bounded Objective Function Method (Reduced Feasible Region Method or Constraint Method).

Utility Function Method (Weighting function method)

As discussed in lecture 1, a multi-objective optimization problem with inequality (or equality) constraints may be formulated as

$$\text{Find} \qquad X = \begin{Bmatrix} x_1 \\ x_2 \\ . \\ . \\ . \\ x_n \end{Bmatrix} \tag{1}$$

which minimizes $f_1(\boldsymbol{X}), f_2(\boldsymbol{X}), \ldots, f_k(\boldsymbol{X})$ (2)

subject to

$$g_j(X) \le 0, \qquad j = 1, 2, \ldots, m \tag{3}$$

In weighting function method, a utility function is defined for each of the objectives according to the relative importance of f_i.. A simple utility function may be defined as $\boldsymbol{\alpha}_i f_i(\boldsymbol{X})$ for i^{th} objective where $\boldsymbol{\alpha}_i$ is a scalar and represents the weight assigned to the corresponding objective. Then the total utility $\boldsymbol{U}$ may be defined as weighted sum of objective functions as below

$$U = \sum_{i=1}^{k} \alpha_i f_i(X), \qquad \boldsymbol{\alpha_i} > 0, \; i = 1, 2, \ldots, k. \tag{4}$$

The solution vector $\boldsymbol{X}$ may be found by maximizing the total utility as defined above with the constraint (3).

Without any loss to generality, it is customary to assume that $\sum_{i=1}^{k} \alpha_i = 1$ although it is not essential. Also $\boldsymbol{\alpha_i}$ values indicate the relative utility of each of the objectives.

Figure 1 represents the decision space for a given set of constraints and utility functions. Here $\boldsymbol{X} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ and two objectives are $f_1(\boldsymbol{X})$ and $f_2(\boldsymbol{X})$ with upper bound constraints* of type (3) as in figure 1.

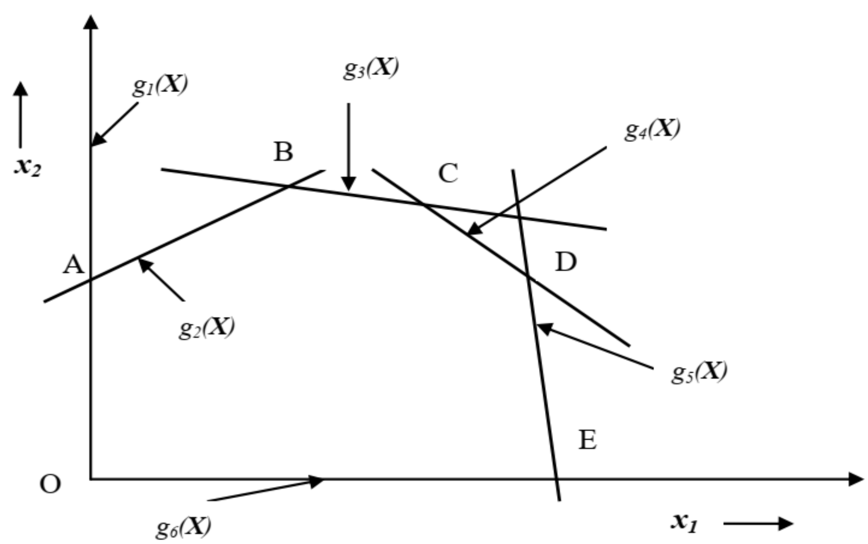


Fig. 1 Decision space

*constraints $g_1(\boldsymbol{X})$ to $g_6(\boldsymbol{X})$ represent $\boldsymbol{x_1}$, $\boldsymbol{x_2} \geq 0$

For Linear Programming (LP), the Pareto front is obtained by plotting the values of objective functions at common points (points of intersection) of constraints and joining them through straight lines in objective space.

It should be noted that all the points on the constraint surface need not be efficient in Pareto sense as point A in figure 2.

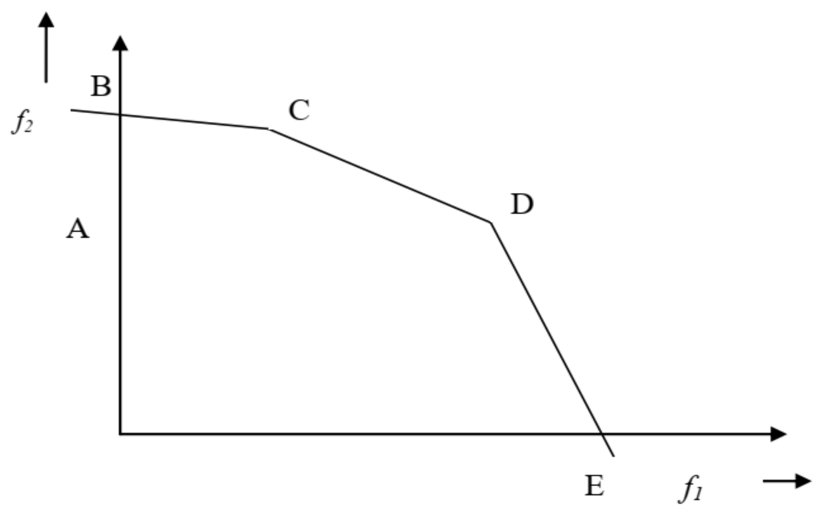


Fig. 2 Objective space

By looking at Figure 2 one may qualitatively infer that it follows Pareto Optimal definition. Now optimizing utility function means moving along the efficient front and looking for the maximum value of utility function U defined by equation (4).

One major limitation is that this method cannot generate the complete set of efficient solutions unless the efficiency frontier is strictly convex. If a part of it is concave, only the end points of this can be obtained by the weighting function method.

Bounded objective function method

In this method we try to trap the optimal solution of the objective functions in a bounded or reduced feasible region. In formulating the problem one objective function is maximized while all other objectives are converted into constraints with lower bounds along with other constraints of the problem. Mathematically the problem may be formulated as

$$\begin{aligned} &\text{Maximize} \quad f_i(\boldsymbol{X}) \\ &\text{Subject to} \quad g_j(X) \le 0, \qquad j = 1, 2, \ldots, m \\ &\qquad f_k(\boldsymbol{X}) \ge e_k \qquad \forall \; k \ne i \end{aligned} \tag{5}$$

here e_k represents lower bound of the kth objective. In this approach the feasible region $\mathcal{S}$ represented by $g_j(X) \le 0$, $j = 1, 2, \ldots, m$ is further reduced to $\mathcal{S}'$ by (k-1) constraints $f_k(\boldsymbol{X}) \ge e_k \;\; \forall \;\; k \ne i$.

For example, let there are three objectives which are to be maximized in the region of constraints $\mathcal{S}$. The problem may be formulated as:

$$\begin{aligned} &\text{Maximize \{objective-1\}} \\ &\text{Maximize \{objective-2\}} \\ &\text{Maximize \{objective-3\}} \end{aligned}$$

$$\text{Subject to} \quad \boldsymbol{X} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in \mathcal{S}$$

In the above problem $\mathcal{S}$ identifies the region given by $g_j(X) \le 0$, $j = 1, 2, \ldots, m$.

In the bounded objective function method, the same problem may be formulated as

$$\begin{aligned} &\text{Maximize \{objective-1\}} \\ &\text{Subject to} \\ &\quad \text{\{objective-2\}} \ge e_1 \\ &\quad \text{\{objective-3\}} \ge e_2 \\ &\quad \boldsymbol{X} \in \mathcal{S} \end{aligned}$$

As may be seen, one of the objectives ({objective-1}) is now the only objective and all other objectives are included as constraints. There are lower bounds specified for other objectives which are the minimum values at least to be attained. Subject to these additional constraints, the objective is maximized. Figure 3 illustrates the scheme.

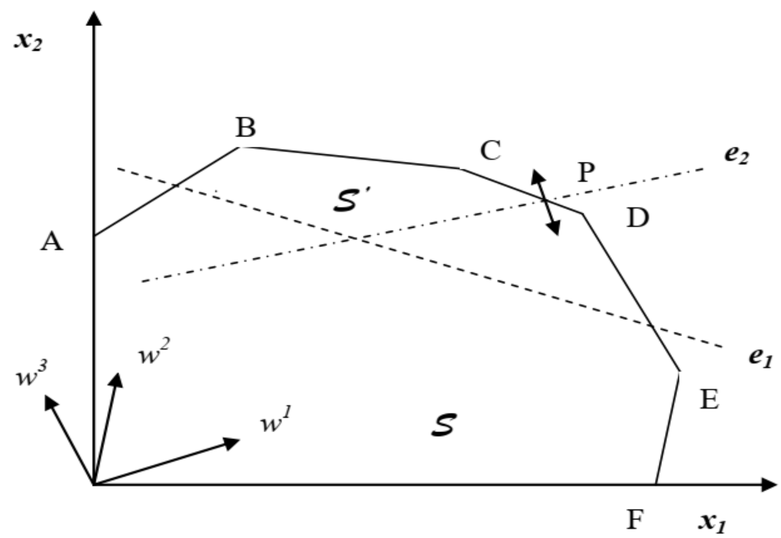


Fig. 3 Bounded objective function

In the above figure w^1, w^2, and w^3 are gradients of the three objectives respectively. If {objective-1} was to be maximized in the region S, without taking into consideration the other objectives, then solution point is E. But due to lower bounds of the other objectives the feasible region reduces to S' and solution point is P now. It may be seen that changing e_1 does not affect {objective-1} as much as changing e_2. This fact gives rise to sensitivity analysis.

Exercise Problem

A reservoir is planned both for gravity and lift irrigation through withdrawals from its storage. The total storage available for both the uses is limited to 5 units each year. It is decided to limit the gravity irrigation withdrawals in a year to 4 units. If X_1 is the allocation of water for gravity irrigation and X_2 is the allocation for lift irrigation, the two objectives planned to be maximized are expressed as

$$\text{Maximize} \quad Z_1(X) = 3x_1 - 2x_2 \quad \text{and} \quad Z_2(X) = -x_1 + 4x_2$$

For above problem, do the following

(i) Generate a Pareto Front of non-inferior (efficient) solutions by plotting Decision space and Objective space.

(ii) Formulate multi objective optimization model using weighting approach with w_1 and w_2 as weights for gravity and lift irrigation respectively.

(iii) Solve it, for (i) w_1=1 and w_2=2 (ii) w_1=2 and w_2=1

(iv) Formulate the problem using constraint method

Solution:

Formulation:

Objective functions:

$$\text{Maximize} \quad Z_1(X) = 3x_1 - 2x_2 \quad \text{and} \quad Z_2(X) = -x_1 + 4x_2$$

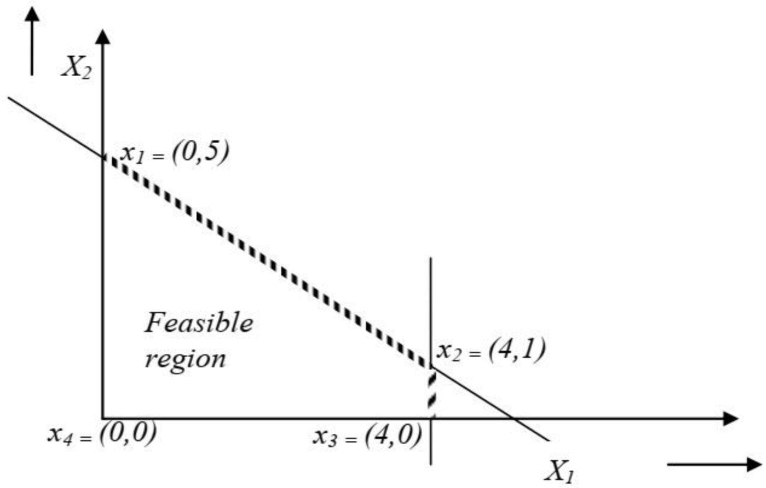
Subject to: $X_1 + X_2 \leq 5;$

$X_1 \leq 5;$

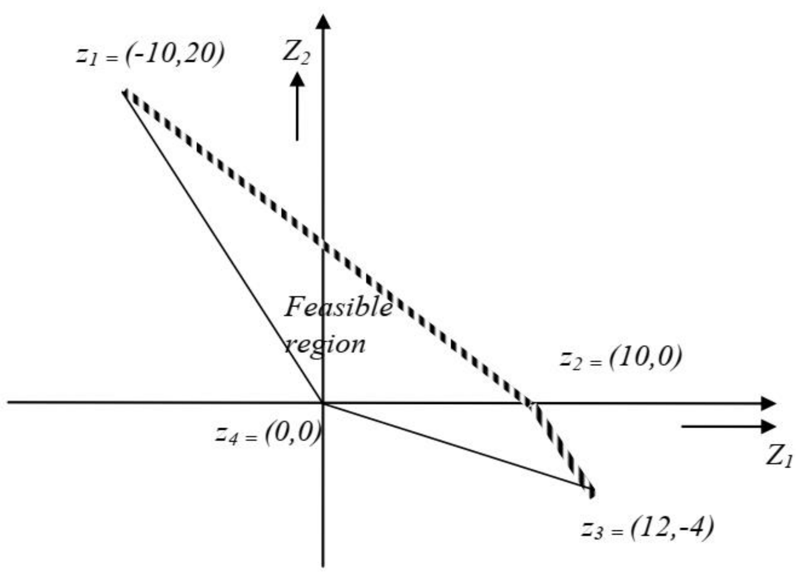
$X_1 \geq 0;$

$X_2 \geq 0.$

(i) Pareto front of non-inferior solutions in decision space



Pareto front of non-inferior solutions in objective space



(ii) Formulation of optimization problem using weighing method

Objective functions:

Maximize $Z = w_1 Z_1 + w_2 Z_2 = w_1 (3x_1 - 2x_2) + w_2 (- x_1 + 4x_2)$

Subject to: $X_1 + X_2 \leq 5$; $X_1 \leq 5$; $X_1 \geq 0$; $X_2 \geq 0$

(iii) (a) w_1=1 and w_2=2

$$Z = (3x_1 - 2x_2) + 2 (- x_1 + 4x_2)$$

$= x_1 + 6x_2$

The Z-line has a slope of -1/6 in decision space and Z has a maximum value of 30 at point (0,5).

(b) w_1=2 and w_2=1

$$Z = 2(3x_1 - 2x_2) + (-x_1 + 4x_2)$$

$$= 5x_1$$

Z has a maximum value of 20 at points (4,1) and (4,0).

(iv) Formulation of optimization problem using constraint method

Maximize $Z_1 = 3x_1 - 2x_2$

Subject to: $Z_2 = (-x_1 + 4x_2) \geq L_2$;

$$X_1 + X_2 \leq 5;$$

$$X_1 \leq 5;$$

$$X_1 \geq 0;$$

$$X_2 \geq 0.$$

Other methods

- Satisficing
- Lexicography
- Indifference Analysis
- Goal Attainment
- Goal-Programming
- Interactive Methods
- Plan Simulation and Evaluation

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