RESERVOIR OPERATION – STEADY STATE OPTIMAL POLICY

Consider a single reservoir receiving inflow i_t and making releases r_t for each time period *t*. The maximum capacity of the reservoir is *K*. The optimization problem is to find the sequence of releases to be made from the reservoir that maximizes the total net benefits. These benefits may be from hydropower generation, irrigation, recreation etc. Let S_t and S_{t+1} be the initial and final storages for time period *t*. Expressing net benefits as a function of S_t, S_{t+1} and r_t, the net benefit for period *t* is $NB_t\left(S_t, S_{t+1}, r_t\right)$.

If there are *T* periods in a year, then the objective function is to maximize the total net benefits from all periods.

$$\text{Maximize } \sum_{t=1}^{T} NB_t\left(S_t, S_{t+1}, r_t\right)$$

This is subject to continuity and also capacity constraints. Neglecting all minor losses like evaporation, seepage etc and assuming that there is no overflow, the continuity relation can be written as,

$$S_{t+1} = S_t + i_t - r_t \qquad for \quad t = 1,2,..T$$

The capacity constraint can be expressed as,

$$S_t \le K \qquad for \quad t = 1,2,..T$$

The above formulated problem can be solved as a sequential process either using forward or backward approach. Here the stages are the time periods and the states are the storage volumes.

Assume that there are *T* periods in a year. In order to find the steady state policy, select a period in a particular year in the near future (to get steady solution). Usually in almost all problems, the last period *T* is taken as the terminal period. At this stage, the optimal release r_t will be independent of the inflow i_t and also the net benefit NB_t.

Now, consider the terminal period as T of a particular year after which reservoir is no longer useful (figure 1). Solving this problem in a backward recursion method, let t represent the period in a year from T to 1 and n represents the periods remaining from t till end. Thus, t will take values starting from T, decreasing to 1 (which will complete one year) and then again taking a value of T and repeating the values. The value of n starts from 1 (while considering the T^{th} period of last year) and while moving backwards its value keeps on increasing i.e. at the beginning of the last year, the value of $n = T$ and at the beginning of second last year its value will be equal to $2T$ and so on.

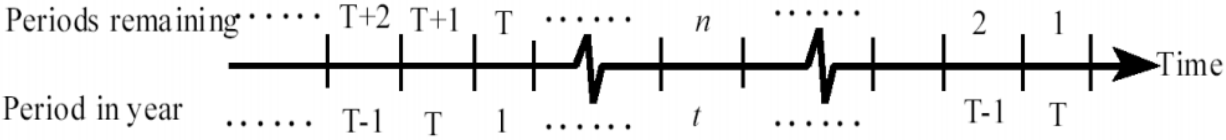

Fig. 1 Sequential process

Starting from T of last year, which is at the far right, there is only one period remaining. Thus, in this case $t=T$ and $n=1$. Let $f_T^1(S_T)$ be the maximum net benefit in the last period of the year considered. $f_T^1(S_T)$ can be expressed as

$$f_T^1(S_T) = \max_{\substack{r_T \geq 0 \\ r_T \leq S_T + i_T \\ r_T \geq S_T + i_T - K}} \left[NB_T\left(S_T, (S_T + i_T - r_T), r_T\right) \right]$$

which should be solved for all S_T values from 0 to K.

Considering the last two stages together for which $t=T-1$ and $n=2$, the objective function can be written as

$$f_{T-1}^2(S_{T-1}) = \max_{\substack{r_{T-1} \geq 0 \\ r_{T-1} \leq S_{T-1} + i_{T-1} \\ r_{T-1} \geq S_{T-1} + i_{T-1} - K}} \left[NB_{T-1}\left(S_{T-1}, (S_{T-1} + i_{T-1} - r_{T-1}), r_{T-1}\right) + f_T^1(S_{T-1} + i_{T-1} - r_{T-1}) \right]$$

This also is solved for all S_{T-1} values from 0 to K.

In general, for a period t of a particular year with n periods remaining, the function can be written as

$$f_t^n(S_t) = \max_{\substack{r_t \ge 0 \\ r_t \le S_t + i_t \\ r_t \ge S_t + i_t - K}} \left[NB_t\left(S_t, S_t + i_t - r_t, r_t\right) + f_{t+1}^{n-1}\left(S_t + i_t - r_t\right) \right]$$

where the index t decreases from T to 1 and then takes the value T again for the previous year and the cycle repeats while the index n starts from 1 and increases at each successive stage.

This cycle can be repeated till the optimum values of r_t for an initial storage S_t will be the same as the corresponding r_t and S_t of previous year. Such a solution is called stationary solution. The maximum net benefit can be obtained as the difference of $f_t^{n+T}(S_t)$ and $f_t^n(S_t)$ for any S_t and t.

Numerical example (Loucks et al., 1981)

Consider a reservoir for which the desirable constant storage is 20 units and the constant release is 25 units. The capacity of the reservoir is 30 units and the inflows for three seasons are given as 10, 50 and 20 units. The problem is to find the optimum S_t and r_t that minimizes the total squared deviation from the release and storage targets given. Hence, the objective function is $\left[(20 - S_t)^2 + (25 - r_t)^2\right]$. Let S_t take the discrete values of 0, 10, 20, 30 and r_t take the values of 10, 20, 30, 40.

Solution:

Consider a year after which the reservoir is no longer useful. The problem can be expressed as a sequential process as shown in the figure 2.

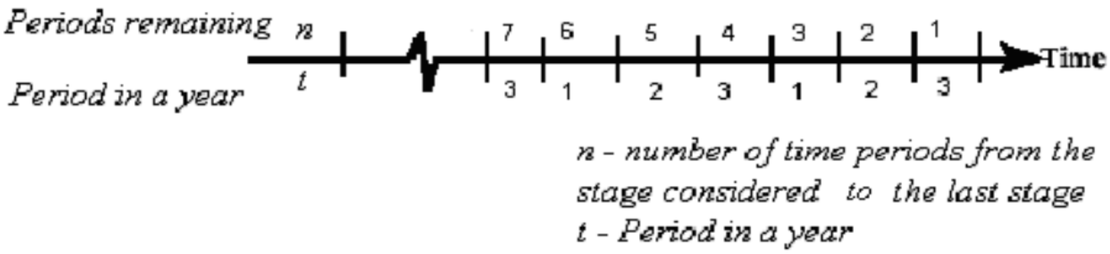

Fig. 2

Here number of seasons (periods), T = 3. Considering the last period for which $t = 3$ and $n = 1$, the optimization function is

$$\text{Minimize } f_3^1(S_3) = \left[(20 - S_3)^2 + (25 - r_3)^2\right]$$

Inflow for 3rd season, $I_3 = 20$ units and capacity of the reservoir, $K = 30$ units.

The release constraints can be expressed as

$$\begin{aligned} r_3 &\le S_3 + I_3 \\ &\le S_3 + 20 \end{aligned} \qquad \text{and}$$

$$\begin{aligned} r_3 &\ge S_3 + I_3 - K \\ &\ge S_3 + 20 - 30 \end{aligned}$$

The computation for the first subproblem (n = 1) is shown in the Table 1.

Table 1

State variable, S_3	Release, r_3	$(20 - S_3)^2 + (25 - r_3)^2$	$f_3^1(S_3)$	Optimal release, r_3^*
0	10	625	425	20
	20	425		
10	10	325	125	20, 30
	20	125		
	30	125		
20	10	225	25	20, 30
	20	25		
	30	25		
	40	225		
30	10	325	125	20, 30
	20	125		
	30	125		
	40	325		

Now considering the last two periods $(n = 2)$, the optimization function is

$$\text{Minimize } f_2^2(S_2) = \left[(20 - S_2)^2 + (25 - r_2)^2\right] + f_3^1(S_2 + I_2 - r_2)$$

Inflow for 2nd season, $I_2 = 50$ units.

The release constraints can be expressed as

$$r_2 \le S_2 + 50 \qquad \text{and}$$

$$r_2 \ge S_2 + 50 - 30$$

The computation for the second subproblem (n = 2) is shown in the table 2.

For $S_2=30$, $r_2 \ge S_2 + 50 - 30$ *i.e.* $r_2 \ge 50$ i.e. $r_2 \ge 50$. Since r_2 can take values only of 10, 20, 30 and 40 only, the release cannot be made for $S_2=30$.

Table 2

State variable, S_2	Release, r_2	$\left[(20-S_2)^2 + (25-r_2)^2\right]$	$S_2+I_2-r_2$	$f_3^1(S_2+I_2-r_2)$	(5)+(3)	$f_2^2(S_2)$	Optimal release, r_2^*
0	20	425	30	125	550	450	30
	30	425	20	25	450		
	40	625	10	125	750		
10	30	125	30	125	250	250	30
	40	325	20	25	350		
20	40	225	30	125	350	350	40
30	*na*	*na*	*na*	*na*	*na*	*na*	*na*

The same procedure is repeated for all stages till *n* = *7*. The summarized solution for this problem is given in the tables 3 to 5.

Table 3

Initial Storage, S_t	*n = 1*		*n = 2*		*n =3*	
	$f_3^1(S_3)$	r_3^*	$f_2^2(S_2)$	r_2^*	$f_1^3(S_1)$	r_1^*
0	425	20	450	30	1075	10
10	125	20, 30	250	30	575	10, 20
20	25	20, 30	350	40	275	20
30	125	20, 30	--	*na*	375	30

Table 4

Initial Storage, S_t	$n = 4$		$n = 5$		$n = 6$	
	$f_3^4(S_3)$	r_3^*	$f_2^5(S_2)$	r_2^*	$f_1^6(S_1)$	r_1^*
0	1200	10	725	30	1350	10
10	600	10	525	30	850	10, 20
20	300	20	625	40	550	20
30	400	30	--	*na*	650	30

Table 5

Initial Storage, S_t	$n = 7$	
	$f_3^7(S_3)$	r_3^*
0	1475	10
10	875	10
20	575	20
30	675	30

At this stage, the value of r_3^* at $n = 7$ and $n = 4$ are exactly the same. Also the difference $f_3^7(S_3) - f_3^4(S_3) = 275$ is same for all S_t. This value is the minimum total squared deviations from the target release and storage.

Thus, the stationary policy obtained is given in Table 6.

Table 6

S_t	Optimal Releases		
	r_1^*	r_2^*	r_3^*
0	10	30	10
10	10, 20	30	10
20	20	40	20
30	30	--	30

A main assumption made in dynamic programming is that the decisions made at one stage is dependent only on the state variable and is independent of the decisions taken in other stages. In cases where decisions made at one stage are dependent on the earlier decisions, then dynamic programming will not be an appropriate optimization technique.

BIBLIOGRAPHY / FURTHER READING:

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