Lecture 33

Learning Objectives

At the end of this class, students should be able to:

- familiar with scalar product of four vectors
- familiar with vector product of four vectors
- familiar with reciprocal system of vectors
- solve related problems

Scalar Product of Four Vectors

Let $\vec{a}, \vec{b}, \vec{c}, \vec{d}$ be any four vectors then $(\vec{a} \times \vec{b}).(\vec{c} \times \vec{d})$ is a scalar quantity called the scalar product of four vectors $\vec{a}, \vec{b}, \vec{c}$ and $\vec{d}$.

Expression for $(\vec{a} \times \vec{b}).(\vec{c} \times \vec{d})$

Let $(\vec{a} \times \vec{b}) = \vec{m}$. Then

$$
\begin{aligned}
(\vec{a} \times \vec{b}).(\vec{c} \times \vec{d}) &= \vec{m}.(\vec{c} \times \vec{d}) \\
&= (\vec{m} \times \vec{c}).\vec{d} \\
&= [(\vec{a} \times \vec{b}) \times \vec{c}].\vec{d} \\
&= -[\vec{c} \times (\vec{a} \times \vec{b})].\vec{d} \\
&= -[(\vec{c}.\vec{b})\vec{a} - (\vec{c}.\vec{a})\vec{b}].\vec{d} \\
&= [(\vec{c}.\vec{a})\vec{b} - (\vec{c}.\vec{b})\vec{a}].\vec{d} \\
&= (\vec{a}.\vec{c})(\vec{b}.\vec{d}) - (\vec{b}.\vec{c})(\vec{a}.\vec{d})
\end{aligned}
$$

Writing above result in the determinant form, we get

$$
(\vec{a} \times \vec{b}).(\vec{c} \times \vec{d}) = \begin{vmatrix} \vec{a}.\vec{c} & \vec{a}.\vec{d} \\ \vec{b}.\vec{c} & \vec{b}.\vec{d} \end{vmatrix}
$$

Vector Product of Four Vectors

Let $\vec{a}, \vec{b}, \vec{c}, \vec{d}$ be any four vectors then the vector product of $(\vec{a} \times \vec{b})$ and $(\vec{c} \times \vec{d})$ is given by $(\vec{a} \times \vec{b}) \times (\vec{c} \times \vec{d})$.

The product $(\vec{a} \times \vec{b}) \times (\vec{c} \times \vec{d})$ is a vector perpendicular to both $(\vec{a} \times \vec{b})$ and $(\vec{c} \times \vec{d})$. Since the product is perpendicular to $\vec{a} \times \vec{b}$, so it lies in the plane of $\vec{a}$ and $\vec{b}$. Again, the product is perpendicular to $\vec{c} \times \vec{d}$, so it lies in the plane of $\vec{c}$ and $\vec{d}$. The two planes intersect in a line and therefore, $(\vec{a} \times \vec{b}) \times (\vec{c} \times \vec{d})$ is a vector parallel to the line of intersection of two planes, one containing $\vec{a}$ and $\vec{b}$ and the other containing $\vec{c}$ and $\vec{d}$.

Expression for $(\vec{a} \times \vec{b}) \times (\vec{c} \times \vec{d})$

(i) Let $\vec{a} \times \vec{b} = \vec{m}$. Then

$$\begin{aligned}(\vec{a} \times \vec{b}) \times (\vec{c} \times \vec{d}) &= \vec{m} \times (\vec{c} \times \vec{d}) \\ &= (\vec{m}.\vec{d})\vec{c} - (\vec{m}.\vec{c})\vec{d} \\ &= [\vec{d}.(\vec{a} \times \vec{b})]\vec{c} - [\vec{c}.(\vec{a} \times \vec{b})]\vec{d} \\ &= [\vec{d}\ \vec{a}\ \vec{b}]\vec{c} - [\vec{c}\ \vec{a}\ \vec{b}]\vec{d} \\ &= [\vec{a}\ \vec{b}\ \vec{d}]\vec{c} - [\vec{a}\ \vec{b}\ \vec{c}\,]\vec{d} \qquad \text{(i)}\end{aligned}$$

(ii) Let $\vec{c} \times \vec{d} = \vec{n}$. Then

$$\begin{aligned}(\vec{a} \times \vec{b}) \times (\vec{c} \times \vec{d}) &= (\vec{a} \times \vec{b}) \times \vec{n} \\ &= -[\vec{n} \times (\vec{a} \times \vec{b})] \\ &= -[(\vec{n}.\vec{b})\vec{a} - (\vec{n}.\vec{a})\vec{b}] \\ &= (\vec{n}.\vec{a})\vec{b} - (\vec{n}.\vec{b})\vec{a} \\ &= (\vec{a}.\vec{n})\vec{b} - (\vec{b}.\vec{n})\vec{a} \\ &= [\vec{a}.(\vec{c} \times \vec{d})]\vec{b} - [\vec{b}.(\vec{c} \times \vec{d})]\vec{a} \\ &= [\vec{a}\ \vec{c}\ \vec{d}]\vec{b} - [\vec{b}\ \vec{c}\ \vec{d}]\vec{a} \qquad \text{(ii)}\end{aligned}$$

Reciprocal System of Vectors

Let $\vec{a}, \vec{b}, \vec{c}$ be a set of non-coplanar vectors so that $[\vec{a}\ \ \vec{b}\ \ \vec{c}] \neq 0$ and let $\vec{a'}, \vec{b'}, \vec{c'}$ be any other set of vectors.

The vectors $\vec{a'} = \frac{\vec{b} \times \vec{c}}{[\vec{a}\ \vec{b}\ \vec{c}\,]}$, $\vec{b'} = \frac{\vec{c} \times \vec{a}}{[\vec{a}\ \vec{b}\ \vec{c}\,]}$, $\vec{c'} = \frac{\vec{a} \times \vec{b}}{[\vec{a}\ \vec{b}\ \vec{c}\,]}$

are perpendicular to the planes containing $\vec{b}$ and $\vec{c}$, $\vec{c}$ and $\vec{a}$ and $\vec{a}$ and $\vec{b}$ respectively; are called reciprocal system of the vectors $\vec{a}, \vec{b}, \vec{c}$.

Properties of Reciprocal System of Vectors

1. If $\vec{a'}, \vec{b'}, \vec{c'}$ are reciprocal system of vectors $\vec{a}, \vec{b}, \vec{c}$. Then $\vec{a}.\vec{a'} = \vec{b}.\vec{b'} = \vec{c}.\vec{c'} = 1$.

Here, $\vec{a}.\vec{a'} = \vec{a}.\frac{\vec{b} \times \vec{c}}{[\vec{a}\ \vec{b}\ \vec{c}\,]} = \frac{\vec{a}.(\vec{b} \times \vec{c})}{[\vec{a}\ \vec{b}\ \vec{c}\,]} = \frac{[\vec{a}\ \vec{b}\ \vec{c}\,]}{[\vec{a}\ \vec{b}\ \vec{c}\,]} = 1$

Similarly, $\vec{b}.\overrightarrow{b'} = 1$ and $\vec{c}.\overrightarrow{c'} = 1$

2. If $\overrightarrow{a'}, \overrightarrow{b'}, \overrightarrow{c'}$ are reciprocal system of vectors $\vec{a}, \vec{b}, \vec{c}$. Then $\vec{a}.\overrightarrow{b'} = \vec{b}.\overrightarrow{c'} = \vec{c}.\overrightarrow{a'} = \vec{b}.\overrightarrow{a'} = \vec{a}.\overrightarrow{c'} = \vec{c}.\overrightarrow{b'} = 0$.

We know that, $\overrightarrow{b'} = \frac{\vec{c}\times\vec{a}}{[\vec{a}\ \vec{b}\ \vec{c}\,]}$

$\therefore \quad \vec{a}.\overrightarrow{b'} = \frac{\vec{a}.(\vec{c}\times\vec{a})}{[\vec{a}\ \vec{b}\ \vec{c}\,]} = \frac{[\vec{a}\ \vec{c}\ \vec{a}]}{[\vec{a}\ \vec{b}\ \vec{c}\,]} = \frac{0}{[\vec{a}\ \vec{b}\ \vec{c}\,]} = 0$

Similarly, other results can be proved.

Illustration

If $\vec{a}, \vec{b}, \vec{c}$ are non-coplanar, then show that $\vec{b}\times\vec{c},\ \ \vec{c}\times\vec{a}$ and $\vec{a}\times\vec{b}$ are also non-coplanar.

Solution

Suppose $\vec{a}, \vec{b}, \vec{c}$ are non-coplanar then $[\vec{a}\ \ \vec{b}\ \ \vec{c}] \neq 0$

Now, $[\vec{b}\times\vec{c}\ \ \ \vec{c}\times\vec{a}\ \ \vec{a}\times\vec{b}] = (\vec{b}\times\vec{c}\,).[(\,\vec{c}\times\vec{a})\times(\vec{a}\times\vec{b})]$ (i)

Here, $(\,\vec{c}\times\vec{a})\times(\vec{a}\times\vec{b}) = [\vec{c}\ \vec{a}\ \vec{b}]\vec{a} - [\vec{c}\ \vec{a}\ \vec{a}]\vec{b}$

$= [\vec{c}\ \vec{a}\ \vec{b}]\vec{a}$

$= [\vec{a}\ \vec{b}\ \vec{c}]\vec{a}$

$\therefore \quad [\vec{b}\times\vec{c}\ \ \ \vec{c}\times\vec{a}\ \ \vec{a}\times\vec{b}] = (\vec{b}\times\vec{c}\,).[\vec{a}\ \vec{b}\ \vec{c}]\vec{a}$

$= [\vec{a}\ \vec{b}\ \vec{c}]\vec{a}.(\vec{b}\times\vec{c}\,)$

$= [\vec{a}\ \vec{b}\ \vec{c}][\vec{a}\ \vec{b}\ \vec{c}]$

$= [\vec{a}\ \vec{b}\ \vec{c}]^2 \neq 0$

Since $[\vec{b}\times\vec{c}\ \ \ \vec{c}\times\vec{a}\ \ \vec{a}\times\vec{b}] \neq 0$, therefore the vectors $\vec{b}\times\vec{c},\ \ \vec{c}\times\vec{a}$ and $\vec{a}\times\vec{b}$ are non-coplanar.

Illustration

Show that $(\vec{a}\times\vec{b})\times(\vec{c}\times\vec{d}) + (\vec{a}\times\vec{c})\times(\vec{d}\times\vec{b}) + (\vec{a}\times\vec{d})\times(\vec{b}\times\vec{c}) = 2[\vec{b}\ \vec{d}\ \vec{c}]\vec{a}$

Solution

Here, $(\vec{a}\times\vec{b})\times(\vec{c}\times\vec{d}) = [\vec{a}\ \vec{b}\ \vec{d}]\,\vec{c} - [\vec{a}\ \vec{b}\ \vec{c}\,]\,\vec{d}$;

$(\vec{a}\times\vec{c})\times(\vec{d}\times\vec{b}) = [\vec{a}\ \vec{c}\ \vec{b}]\,\vec{d} - [\vec{a}\ \vec{c}\ \vec{d}]\,\vec{b}$

$(\vec{a}\times\vec{d})\times(\vec{b}\times\vec{c}) = [\vec{a}\ \vec{d}\ \vec{c}]\,\vec{b} - [\vec{a}\ \vec{d}\ \vec{b}]\,\vec{c}$

Adding all, we get

$$(\vec{a}\times\vec{b})\times(\vec{c}\times\vec{d})+(\vec{a}\times\vec{c})\times(\vec{d}\times\vec{b})+(\vec{a}\times\vec{d})\times(\vec{b}\times\vec{c})$$

$$= [\vec{a}\ \vec{b}\ \vec{d}]\vec{c} - [\vec{a}\ \vec{b}\ \vec{c}\ ]\vec{d} + [\vec{a}\ \vec{c}\ \vec{b}]\vec{d} - [\vec{a}\ \vec{c}\ \vec{d}]\vec{b} + [\vec{a}\ \vec{d}\ \vec{c}]\vec{b} - [\vec{a}\ \vec{d}\ \vec{b}]\vec{c}$$

$$= [\vec{a}\ \vec{b}\ \vec{d}]\vec{c} - [\vec{a}\ \vec{b}\ \vec{c}\ ]\vec{d} - [\vec{a}\ \vec{b}\ \vec{c}\ ]\vec{d} - [\vec{a}\ \vec{c}\ \vec{d}]\vec{b} - [\vec{a}\ \vec{c}\ \vec{d}]\vec{b} + [\vec{a}\ \vec{b}\ \vec{d}]\vec{c}$$

$$= 2\ [\vec{a}\ \vec{b}\ \vec{d}]\vec{c} - 2[\vec{a}\ \vec{b}\ \vec{c}\ ]\vec{d} - 2[\vec{a}\ \vec{c}\ \vec{d}]\vec{b}$$

$$= 2\ \{[\vec{a}\ \vec{b}\ \vec{d}]\vec{c} - [\vec{a}\ \vec{b}\ \vec{c}\ ]\vec{d}\} - 2[\vec{a}\ \vec{c}\ \vec{d}]\vec{b}$$

$$= 2\ \{(\vec{a}\times\vec{b})\times(\vec{c}\times\vec{d})\} - 2\left[\vec{a}\,\vec{c}\,\vec{d}\right]\vec{b}$$

$$= 2\ \{(\vec{c}\times\vec{d})\times(\vec{b}\times\vec{a})\} - 2\left[\vec{a}\,\vec{c}\,\vec{d}\right]\vec{b} \qquad [\because \vec{a}\times\vec{b} = -\vec{b}\times\vec{a}]$$

$$= 2\ [\vec{c}\ \vec{d}\ \vec{a}\ ]\vec{b} - 2[\vec{c}\ \vec{d}\ \vec{b}]\vec{a} - 2[\vec{a}\ \vec{c}\ \vec{d}]\vec{b}$$

$$= 2\ [\vec{a}\ \vec{c}\ \vec{d}]\vec{b} - 2[\vec{c}\ \vec{d}\ \vec{b}]\vec{a} - 2[\vec{a}\ \vec{c}\ \vec{d}]\vec{b}$$

$$= -2\left[\vec{c}\,\vec{d}\,\vec{b}\right]\vec{a}$$

$$= -2\left[\vec{d}\,\vec{b}\,\vec{c}\right]\vec{a}$$

$$= 2\left[\vec{b}\,\vec{d}\ \ \vec{c}\right]\vec{a}$$

Illustration

Show that $(\vec{b}\times\vec{c}).(\vec{a}\times\vec{d})+(\vec{c}\times\vec{a}).(\vec{b}\times\vec{d})+(\vec{a}\times\vec{b}).(\vec{c}\times\vec{d}) = 0$

Solution

We know that

$$(\vec{b}\times\vec{c}).(\vec{a}\times\vec{d}) = \begin{vmatrix} \vec{b}.\vec{a} & \vec{b}.\vec{d} \\ \vec{c}.\vec{a} & \vec{c}.\vec{d} \end{vmatrix}$$

$$= (\vec{b}.\vec{a})(\vec{c}.\vec{d}) - (\vec{c}.\vec{a})(\vec{b}.\vec{d})$$

Similarly,

$$(\vec{c}\times\vec{a}).(\vec{b}\times\vec{d}) = (\vec{c}.\vec{b})(\vec{a}.\vec{d}) - (\vec{a}.\vec{b})(\vec{c}.\vec{d}) \text{ and}$$

$$(\vec{a}\times\vec{b}).(\vec{c}\times\vec{d}) = (\vec{a}.\vec{c})(\vec{b}.\vec{d}) - (\vec{b}.\vec{c})(\vec{a}.\vec{d})$$

Now,

$$(\vec{b}\times\vec{c}).(\vec{a}\times\vec{d})+(\vec{c}\times\vec{a}).(\vec{b}\times\vec{d})+(\vec{a}\times\vec{b}).(\vec{c}\times\vec{d})$$

$$= (\vec{b}.\vec{a})(\vec{c}.\vec{d}) - (\vec{c}.\vec{a})(\vec{b}.\vec{d}) + (\vec{c}.\vec{b})(\vec{a}.\vec{d}) - (\vec{a}.\vec{b})(\vec{c}.\vec{d}) + (\vec{a}.\vec{c})(\vec{b}.\vec{d}) - (\vec{b}.\vec{c})(\vec{a}.\vec{d})$$

$$= 0$$

Exercise for Reader

1. If $\vec{a}, \vec{b}, \vec{c}$ are coplanar, then show that $\vec{b} \times \vec{c}$, $\vec{c} \times \vec{a}$ and $\vec{a} \times \vec{b}$ are also coplanar.
2. Show that $(\vec{a} \times \vec{b}) \times (\vec{c} \times \vec{d}) + (\vec{b} \times \vec{c}) \times (\vec{a} \times \vec{d}) + (\vec{c} \times \vec{a}) \times (\vec{b} \times \vec{d}) = -2[\vec{a}\ \vec{b}\ \vec{c}]\vec{d}$
3. Show that $\{(\vec{a} + \vec{b} + \vec{c}) \times (\vec{b} + \vec{c})\}.\vec{c} = \vec{a}.(\vec{b} \times \vec{c})$
4. Show that $[\vec{l}\ \vec{m}\ \vec{n}][\vec{a}\ \vec{b}\ \vec{c}] = \begin{vmatrix} \vec{l}.\vec{a} & \vec{l}.\vec{b} & \vec{l}.\vec{c} \\ \vec{m}.\vec{a} & \vec{m}.\vec{b} & \vec{m}.\vec{c} \\ \vec{n}.\vec{a} & \vec{n}.\vec{b} & \vec{n}.\vec{c} \end{vmatrix}$
5. If $\vec{a}, \vec{b}, \vec{c}$ and $\overrightarrow{a'}, \overrightarrow{b'}, \overrightarrow{c'}$ are reciprocal system of vectors, show that $\vec{a} \times \overrightarrow{a'} + \vec{b} \times \overrightarrow{b'} + \vec{c} \times \overrightarrow{c'} = 0$.