Lecture 32

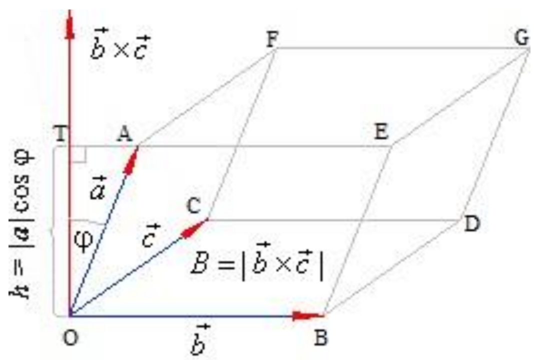
Learning Objectives

At the end of this class, students should be able to:

- understand the concept of scalar triple product
- understand the concept of vector triple product
- solve related problems

Scalar Triple Product

Let $\vec{a}, \vec{b}, \vec{c}$ be three vectors. The scalar product of $\vec{a}$ and $\vec{b} \times \vec{c}$ is a scalar. It is written as $\vec{a}.(\vec{b} \times \vec{c})$ or $[\vec{a}\ \vec{b}\ \vec{c}]$ or $[\vec{a}, \vec{b}, \vec{c}]$.



Let us consider a parallelepiped with three concurrent edges OA, OB, and OC which represent three vectors $\vec{a}, \vec{b}$ and $\vec{c}$ respectively. We know that $\vec{b} \times \vec{c}$ is the vector perpendicular to the plane containing $\vec{b}$ and $\vec{c}$. The area of parallelogram OBDC is denoted by $|\vec{b} \times \vec{c}\,|$.

Now, $\vec{a}.(\vec{b} \times \vec{c}) = |\vec{b} \times \vec{c}\,| \times$ Projection of $\vec{a}$ on $\vec{b} \times \vec{c}$

= (Area of ▱OBDC) × OT

= (Area of ▱OBDC) × (height of the parallelepiped)

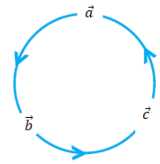
= Volume of the parallelepiped

Thus, $V = \vec{a}.(\vec{b} \times \vec{c})$

Since the scalar triple product $\vec{a}.(\vec{b} \times \vec{c})$ represents the volume of a box, i.e., parallelepiped, it is so called box product and so it is denoted by $[\vec{a}\ \vec{b}\ \vec{c}]$ or $[\vec{a}, \vec{b}, \vec{c}]$, and read as "box $\vec{a}\ \vec{b}\ \vec{c}$".

Note:

1. The scalar product remains same as long as the cyclic order of the vectors $\vec{a}, \vec{b}, \vec{c}$ is maintained.



$$\vec{a}.(\vec{b} \times \vec{c}) = \vec{b}.(\vec{c} \times \vec{a}) = \vec{c}.(\vec{a} \times \vec{b})$$

or, $$(\vec{b} \times \vec{c}).\vec{a} = (\vec{c} \times \vec{a}).\vec{b} = (\vec{a} \times \vec{b}).\vec{c}$$

2. $\vec{a}, \vec{b}, \vec{c}$ are in anticlockwise direction, and $\vec{a}, \vec{c}, \vec{b}$ in clockwise direction.

Now, $[\vec{a}\ \vec{c}\ \vec{b}] = \vec{a}.(\vec{c} \times \vec{b}) = \vec{a}.\{-(\vec{b} \times \vec{c})\}$
$= -\vec{a}.(\vec{b} \times \vec{c}) = -[\vec{a}\ \vec{b}\ \vec{c}]$

3. If the cyclic order of the vectors is reversed, then

$$\vec{b}.(\vec{a} \times \vec{c}) = \vec{a}.(\vec{c} \times \vec{b}) = \vec{c}.(\vec{b} \times \vec{a})$$

Some Important Results

1. In a scalar triple product, the dot and cross can be interchanged without changing the value of the result, i.e., $\vec{a}.(\vec{b} \times \vec{c}) = (\vec{a} \times \vec{b}).\vec{c}$.

Proof: $(\vec{a} \times \vec{b}).\vec{c} = \vec{c}.(\vec{a} \times \vec{b})$
$= [\vec{c}\ \vec{a}\ \vec{b}]$
$= [\vec{a}\ \vec{b}\ \vec{c}]$
$= \vec{a}.(\vec{b} \times \vec{c})$

2. A necessary and sufficient condition that the three non-zero vectors $\vec{a}, \vec{b}, \vec{c}$ be coplanar is $[\vec{a}\ \vec{b}\ \vec{c}] = 0$.

Proof: Let the vectors $\vec{a}, \vec{b}, \vec{c}$ be coplanar. The vector $\vec{b} \times \vec{c}$ is perpendicular to the plane of $\vec{b}$ and $\vec{c}$ Since $\vec{a}$ lies in the same plane, so $\vec{b} \times \vec{c}$ is perpendicular to $\vec{a}$. Therefore

$$\vec{a}.(\vec{b} \times \vec{c}) = 0$$

$\therefore$ $$[\vec{a}\ \vec{b}\ \vec{c}] = 0$$

Conversely: Let $\vec{a}.(\vec{b} \times \vec{c}) = 0$

Then $\vec{b} \times \vec{c}$ is perpendicular to $\vec{a}$. Also, $\vec{b} \times \vec{c}$ is perpendicular to the plane containing $\vec{b}$ and $\vec{c}$. That is, $\vec{a}$ lies in the plane containing $\vec{b}$ and $\vec{c}$. Hence, the vectors $\vec{a}, \vec{b}$ and $\vec{c}$ are coplanar.

3. When two vectors are equal then $[\vec{a}\ \vec{b}\ \vec{c}] = 0$.

Proof: If $\vec{a} = \vec{b}$, then

$\vec{a}.(\vec{b} \times \vec{c}) = \vec{a}.(\vec{a} \times \vec{c})$
$= (\vec{a} \times \vec{a}).\vec{c}$ $[\because \vec{a} \times \vec{a} = 0]$
$= 0$

4. When two vectors are parallel then $[\vec{a}\ \vec{b}\ \vec{c}] = 0$.

Proof: Let $\vec{b}$ and $\vec{c}$ are parallel, then $\vec{b} = \lambda\vec{c}$

Now, $\vec{a}.(\vec{b} \times \vec{c}) = \vec{a}.(\lambda\vec{c} \times \vec{c})$

$= \lambda\vec{a}.(\vec{c} \times \vec{c})$

$= \lambda\vec{a}.\vec{o}$

$= 0$

5. Scalar triple product is distributive, i.e., $[\vec{a}\ \ \vec{b} + \vec{c}\ \ \vec{d}] = [\vec{a}\,\vec{b}\,\vec{d}] + [\vec{a}\,\vec{c}\,\vec{d}]$.

Proof: Here, $[\vec{a}\ \ \vec{b} + \vec{c}\ \ \vec{d}] = \{\vec{a} \times (\vec{b} + \vec{c})\}.\vec{d}$

$= \{\vec{a} \times \vec{b} + \vec{a} \times \vec{c})\}.\vec{d}$

$= (\vec{a} \times \vec{b}).\vec{d} + (\vec{a} \times \vec{c}).\vec{d}$

$= [\vec{a}\,\vec{b}\,\vec{d}] + [\vec{a}\,\vec{c}\,\vec{d}]$.

Scalar Triple Product in Determinant Form

Let $\vec{a} = a_1\vec{i} + a_2\vec{j} + a_3\vec{k}$, $\vec{b} = b_1\vec{i} + b_2\vec{j} + b_3\vec{k}$, and $\vec{c} = c_1\vec{i} + c_2\vec{j} + c_3\vec{k}$

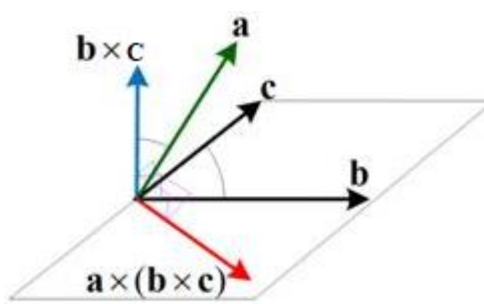
Now, $\vec{b} \times \vec{c} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$

$= (b_2c_3 - b_3c_2)\vec{i} + (b_3c_1 - b_1c_3)\vec{j} + (b_1c_2 - b_2c_1)\vec{k}$

So, $\vec{a}.(\vec{b} \times \vec{c}) = a_1(b_2c_3 - b_3c_2) + a_2(b_3c_1 - b_1c_3) + a_3(b_1c_2 - b_2c_1)$

$= \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$

Vector Triple Product



Let $\vec{a}, \vec{b}, \vec{c}$ be any three vectors. The vector product of $\vec{a}$ and $\vec{b} \times \vec{c}$ is denoted by $\vec{a} \times (\vec{b} \times \vec{c})$. This vector product $\vec{a} \times (\vec{b} \times \vec{c})$ is a vector perpendicular to both $\vec{a}$ and $(\vec{b} \times \vec{c})$. But $\vec{b} \times \vec{c}$ is a vector perpendicular to $\vec{b}$ **and** $\vec{c}$.

Thus $\vec{a} \times (\vec{b} \times \vec{c})$ is a vector coplanar with $\vec{b}$ **and** $\vec{c}$. Moreover, $\vec{a} \times (\vec{b} \times \vec{c})$ is a vector which is coplanar with $\vec{b}$ **and** $\vec{c}$ and perpendicular to $\vec{a}$.

The following is the expression for $\vec{a} \times (\vec{b} \times \vec{c})$:

$$\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a}.\vec{c})\vec{b} - (\vec{a}.\vec{b})\vec{c}$$

Note: $\vec{a} \times (\vec{b} \times \vec{c}) \neq (\vec{a} \times \vec{b}) \times \vec{c}$ in general.

Illustration

Find the volume of parallelepiped whose adjacent edges are represented by the vectors: $\vec{i}+\vec{j}+\vec{k}$, $\vec{i}-\vec{j}+\vec{k}$, and $\vec{i}+2\vec{j}-\vec{k}$.

Solution

Let $\vec{a}=\vec{i}+\vec{j}+\vec{k},\ \vec{b}=\vec{i}-\vec{j}+\vec{k},$ and $\vec{c}=\vec{i}+2\vec{j}-\vec{k}$. If v represents the volume of parallelepiped, then

$$v = \vec{a}.\left(\vec{b}\times\vec{c}\right)$$

Here, $\vec{b}\times\vec{c} = \begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ 1 & -1 & 1\\ 1 & 2 & -1\end{vmatrix}$

$$= (1-2)\vec{i}-(-1-1)\vec{j}+(2+1)\vec{k}$$

$$= -\vec{i}+2\vec{j}+3\vec{k}$$

Now, $\vec{a}.\left(\vec{b}\times\vec{c}\right)=\left(\vec{i}+\vec{j}+\vec{k}\right).\left(-\vec{i}+2\vec{j}+3\vec{k}\right)$

$$= -1+2+3=4$$

Illustration

Prove that the following four points with the position vectors: $2\vec{i}+3\vec{j}-\vec{k}, \vec{i}-2\vec{j}+3\vec{k}, 3\vec{i}+4\vec{j}-2\vec{k}$ and $\vec{i}-6\vec{j}+6\vec{k}$ respectively are coplanar.

Solution

Let the given points be A, B, C, and D respectively. Let O be the origin. Then

$\overrightarrow{OA}=2\vec{i}+3\vec{j}-\vec{k},\ \overrightarrow{OB}=\vec{i}-2\vec{j}+3\vec{k},\ \overrightarrow{OC}=3\vec{i}+4\vec{j}-2\vec{k}$ and $\overrightarrow{OD}=\vec{i}-6\vec{j}+6\vec{k}$. Then

$\overrightarrow{AB}=\overrightarrow{OB}-\overrightarrow{OA}=-\vec{i}-5\vec{j}+4\vec{k}$

$\overrightarrow{BC}=\overrightarrow{OC}-\overrightarrow{OB}=2\vec{i}+6\vec{j}-5\vec{k}$

$\overrightarrow{CD}=\overrightarrow{OD}-\overrightarrow{OC}=-2\vec{i}-10\vec{j}+8\vec{k}$

If the points A, B, C, and D lie on the same plane, then the vectors $\overrightarrow{AB}, \overrightarrow{BC}, \overrightarrow{CD}$ are coplanar.

If the vectors $\overrightarrow{AB}, \overrightarrow{BC}, \overrightarrow{CD}$ are coplanar then the scalar triple product of $\overrightarrow{AB}, \overrightarrow{BC}, \overrightarrow{CD}$ should be zero.

i.e., $\left[\overrightarrow{AB}\ \overrightarrow{BC}\ \overrightarrow{CD}\right]=0 \quad \Rightarrow \overrightarrow{AB}.\left(\overrightarrow{BC}\times\overrightarrow{CD}\right)=0$

Here, $\overrightarrow{BC}\times\overrightarrow{CD} = \begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ 2 & 6 & -5\\ -2 & -10 & 8\end{vmatrix}$

$$= 98\vec{i}-26\vec{j}-8\vec{k}$$

Now, $\overrightarrow{AB}.\left(\overrightarrow{BC}\times\overrightarrow{CD}\right)=\left(-\vec{\imath}-5\vec{\jmath}+4\vec{k}\right).(98\vec{\imath}-26\vec{\jmath}-8\vec{k})$

$$= -98+130-32$$
$$= 0$$

Hence, the given points are coplanar.

Illustration

Find the volume of tetrahedron whose vertices are P(3, 4, 5), A (2,1,1), B(2,1,5) and C(1, 4,2).

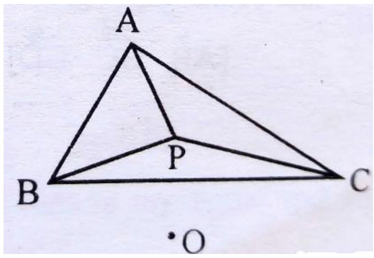
Solution

Let ABC be a tetrahedron with edges PA, PB and PC. If v represents the volume of the tetrahedron, then

$$V=\frac{1}{6}\left[\overrightarrow{PA}\ \overrightarrow{PB}\ \overrightarrow{PC}\right]$$

Let O be the origin. Then

$\overrightarrow{OP}=3\vec{\imath}+4\vec{\jmath}+5\vec{k},\ \overrightarrow{OA}=2\vec{\imath}+\vec{\jmath}+\vec{k},\ \overrightarrow{OB}=2\vec{\imath}+\vec{\jmath}+5\vec{k}$ and $\overrightarrow{OC}=\vec{\imath}+4\vec{\jmath}+2\vec{k}$. Then



$\overrightarrow{PA}=\overrightarrow{OA}-\overrightarrow{OP}=-\vec{\imath}-3\vec{\jmath}-4\vec{k}$

$\overrightarrow{PB}=\overrightarrow{OB}-\overrightarrow{OP}=-\vec{\imath}-3\vec{\jmath}$

$\overrightarrow{PC}=\overrightarrow{OC}-\overrightarrow{OP}=-2\vec{\imath}-3\vec{k}$

Here, $\overrightarrow{PB}\times\overrightarrow{PC}=\begin{vmatrix}\vec{\imath} & \vec{\jmath} & \vec{k}\\ -1 & -3 & 0\\ -2 & 0 & -3\end{vmatrix}$

$$= 9\vec{\imath}-3\vec{\jmath}-6\vec{k}$$

Now, $\overrightarrow{PA}.\left(\overrightarrow{PB}\times\overrightarrow{PC}\right)=\left(-\vec{\imath}-3\vec{\jmath}-4\vec{k}\right).(9\vec{\imath}-3\vec{\jmath}-6\vec{k})$

$$= -9+9+24=24$$

$\therefore\quad V=\frac{1}{6}\left[\overrightarrow{PA}\ \overrightarrow{PB}\ \overrightarrow{PC}\right]=\frac{1}{6}\times\overrightarrow{PA}.\left(\overrightarrow{PB}\times\overrightarrow{PC}\right)$

$$=\frac{1}{6}\times 24$$
$$= 4$$

Illustration

Show that $\left[\vec{a}+\vec{b}\ \ \vec{b}+\vec{c}\ \ \vec{c}+\vec{a}\right]=2\left[\vec{a}\ \vec{b}\ \vec{c}\right]$

Solution

LHS: $\left[\vec{a}+\vec{b}\ \ \vec{b}+\vec{c}\ \ \vec{c}+\vec{a}\right]$

$$=\left(\vec{a}+\vec{b}\right).\left[(\vec{b}+\vec{c})\times(\vec{c}+\vec{a})\right]$$

$= (\vec{a}+\vec{b}).\left[\vec{b}\times\vec{c}+\vec{b}\times\vec{a}+\vec{c}\times\vec{c}+\vec{c}\times\vec{a}\right]$

$= (\vec{a}+\vec{b}).\left[\vec{b}\times\vec{c}+\vec{b}\times\vec{a}+\vec{c}\times\vec{a}\right]$ $\qquad [\because \vec{c}\times\vec{c}=0]$

$= \vec{a}.(\vec{b}\times\vec{c})+\vec{b}.(\vec{b}\times\vec{c})+\vec{a}.(\vec{b}\times\vec{a})+\vec{b}.(\vec{b}\times\vec{a})+\vec{a}.(\vec{c}\times\vec{a})+\vec{b}.(\vec{c}\times\vec{a})$

$= [\vec{a}\ \vec{b}\ \vec{c}]+[\vec{b}\ \vec{b}\ \vec{c}]+[\vec{a}\ \vec{b}\ \vec{a}]+[\vec{b}\ \vec{b}\ \vec{a}]+[\vec{a}\ \vec{c}\ \vec{a}]+[\vec{b}\ \vec{c}\ \vec{a}]$

$= [\vec{a}\ \vec{b}\ \vec{c}]+0+0+0+0+[\vec{a}\ \vec{b}\ \vec{c}]$

[∵ scalar triple product is zero when two vectors are equal]

$= 2[\vec{a}\ \vec{b}\ \vec{c}]$

Illustration

Show that $\vec{a}\times(\vec{b}\times\vec{c})+\vec{b}\times(\vec{c}\times\vec{a})+\vec{c}\times(\vec{a}\times\vec{b})=0$

Solution

LHS: $\vec{a}\times(\vec{b}\times\vec{c})+\vec{b}\times(\vec{c}\times\vec{a})+\vec{c}\times(\vec{a}\times\vec{b})$

$= (\vec{a}.\vec{c})\vec{b}-(\vec{a}.\vec{b})\vec{c}+(\vec{b}.\vec{a})\vec{c}-(\vec{b}.\vec{c})\vec{a}+(\vec{c}.\vec{b})\vec{a}-(\vec{c}.\vec{a})\vec{b}$

$= (\vec{a}.\vec{c})\vec{b}-(\vec{a}.\vec{b})\vec{c}+(\vec{a}.\vec{b})\vec{c}-(\vec{b}.\vec{c})\vec{a}+(\vec{b}.\vec{c})\vec{a}-(\vec{a}.\vec{c})\vec{b}$

$= 0$

Exercise for Reader

1. Find the volume of parallelepiped whose adjacent edges are represented by the vectors: $2\vec{i}-3\vec{j}+4\vec{k}$, $\vec{i}+2\vec{j}-\vec{k}$, and $3\vec{i}-\vec{j}+2\vec{k}$.
2. Prove that the following four points with the position vectors: $4\vec{i}+5\vec{j}+\vec{k}$, $-\vec{j}-\vec{k}$, $3\vec{i}+9\vec{j}+4\vec{k}$ and $-4\vec{i}+4\vec{j}+4\vec{k}$ respectively are coplanar.
3. Find the constant p so that the vectors $\vec{a}=2\vec{i}-3\vec{j}+\vec{k}$, $\vec{b}=\vec{i}+2\vec{j}+3\vec{k}$, and $\vec{c}=3\vec{i}+p\vec{j}+5\vec{k}$ are coplanar.
4. If $\vec{a},\vec{b},\vec{c}$ are non-coplanar, then prove that the vectors $\vec{a}+\vec{b}$, $\vec{b}+\vec{c}$, $\vec{c}+\vec{a}$ are non-coplanar.
5. If $\vec{a},\vec{b},\vec{c}$ are coplanar, then prove that the vectors $\vec{a}+\vec{b}$, $\vec{b}+\vec{c}$, $\vec{c}+\vec{a}$ are coplanar.
6. If $\vec{a}=2\vec{i}+3\vec{j}+4\vec{k}$, $\vec{b}=\vec{i}+\vec{j}-\vec{k}$, and $\vec{c}=\vec{i}-\vec{j}+\vec{k}$ find $\vec{a}\times(\vec{b}\times\vec{c})$ and verify that $\vec{a}\times(\vec{b}\times\vec{c})$ is perpendicular to $\vec{a}$ and $\vec{b}\times\vec{c}$.