

Lecture 31

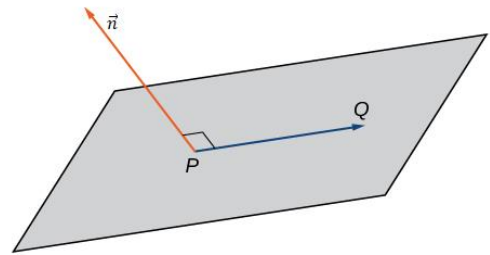
Learning Objectives

At the end of this class, students should be able to:

- understand the concept of vector equation of plane
- identify the equation of plane passing through a given point
- identify the equation of plane passing through three given points
- solve related problems

Plane

Let \vec{n} be a vector and $P(x_1, y_1, z_1)$ be the given point. Then the set of all points $Q(x, y, z)$ such that \vec{PQ} is perpendicular to \vec{n} forms a plane. We say that \vec{n} is a normal vector, or perpendicular to the plane. Remember, the dot product of orthogonal vectors is zero. This fact generates the vector equation of a plane: $\vec{n} \cdot \vec{PQ} = 0$.

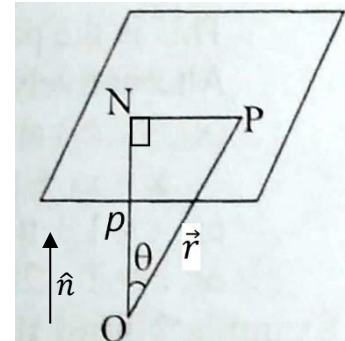


Thus, given a point P and vector \vec{n} , the set of all points Q satisfying the equation $\vec{n} \cdot \vec{PQ} = 0$ forms a plane.

Vector Equation of a Plane

If \hat{n} is a unit vector perpendicular to a plane, and p is the length of perpendicular from the origin on the plane, then the vector equation of plane is $\vec{r} \cdot \hat{n} = p$

Proof: Let O be the origin and let ON be the perpendicular drawn from the origin to the plane. Let \hat{n} be the unit vector along the direction of ON . Let \vec{r} be the position vector of any point P in the plane. Joining OP and NP .



$$\begin{aligned} \text{Now, } \vec{r} \cdot \hat{n} &= |\vec{r}| \cdot |\hat{n}| \cos \theta \\ &= |\vec{r}| \cos \theta && [\because |\hat{n}| = 1] \\ &= ON && \left[\because \cos \theta = \frac{ON}{|\vec{r}|} \right] \end{aligned}$$

$$\therefore \vec{r} \cdot \hat{n} = p$$

Since P is any point on the plane, so $\vec{r} \cdot \hat{n} = p$ represents a vector equation of the plane.

Illustration

The perpendicular distance of a plane from the origin is 3 units and the vector $2\vec{i} + 2\vec{j} - \vec{k}$ is a normal to the plane. Find the equation of the plane.

Solution

Let $\vec{v} = 2\vec{i} + 2\vec{j} - \vec{k}$ and $p = 3$.

Let \hat{n} be the unit normal vector perpendicular to the plane along \vec{v} . Then

$$\hat{n} = \frac{\vec{v}}{|\vec{v}|} = \frac{2\vec{i} + 2\vec{j} - \vec{k}}{3}$$

Let $P(x, y, z)$ be any point on the plane whose position vector is \vec{r} . Then the equation of the plane is $\vec{r} \cdot \hat{n} = p$

or, $(x\vec{i} + y\vec{j} + z\vec{k}) \cdot \frac{1}{3}(2\vec{i} + 2\vec{j} - \vec{k}) = 3$

or, $2x + 2y - 2z = 9$

Which is the required equation of plane.

Plane Through a Given Point

Let O be the origin and A be the given point on the plane whose position vector is \vec{a} . Let P be any point on the plane whose position vector is \vec{r} . Let \vec{n} be any vector perpendicular to the plane.

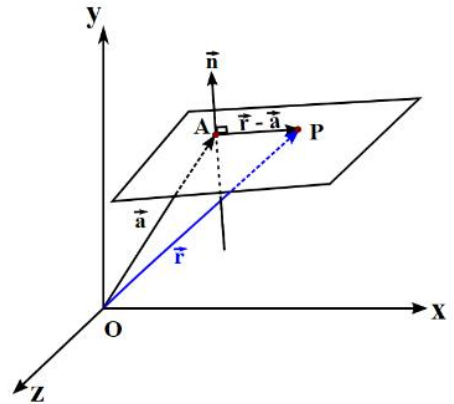
Since AP lies in the plane, so \vec{AP} is perpendicular to \vec{n} . Therefore,

$$\vec{AP} \cdot \vec{n} = 0$$

i.e., $(\vec{OP} - \vec{OA}) \cdot \vec{n} = 0$

or, $(\vec{r} - \vec{a}) \cdot \vec{n} = 0$

Since P is any point on the plane, so $(\vec{r} - \vec{a}) \cdot \vec{n} = 0$ represents the equation of plane normal to \vec{n} and passing through the point A.



Illustration

Find the equation of the plane passing through the point $\vec{i} + \vec{j} + \vec{k}$ and perpendicular to the vector $2\vec{i} + 3\vec{j} - 4\vec{k}$.

Solution

Let $P(x, y, z)$ be any point on the plane whose position vector is \vec{r} . Let $\vec{a} = \vec{i} + \vec{j} + \vec{k}$ and $\vec{n} = 2\vec{i} + 3\vec{j} - 4\vec{k}$. Then the equation of the plane that passes through \vec{a} and perpendicular to \vec{n} is

$$(\vec{r} - \vec{a}) \cdot \vec{n} = 0$$

i.e., $\{(x\vec{i} + y\vec{j} + z\vec{k}) - (\vec{i} + \vec{j} + \vec{k})\} \cdot (2\vec{i} + 3\vec{j} - 4\vec{k}) = 0$

$$\text{or, } (x - 1) \times 2 + (y - 1) \times 3 + (z - 1) \times (-4) = 0$$

$$\text{or, } 2x + 3y - 4z = 1$$

Which is the required equation of plane.

Illustration

Find the equation of the plane passing through the point $(1, -1, 3)$ and parallel to the plane $3x + y + z = 7$.

Solution

The equation of the given plane is $3x + y + z = 7$ which can be written as

$$\vec{r} \cdot (3\vec{i} + \vec{j} + \vec{k}) = 7 \text{ which is of the form } \vec{r} \cdot \vec{n} = D$$

Thus, the normal vector to the given plane is $\vec{n} = 3\vec{i} + \vec{j} + \vec{k}$.

Since, another plane is parallel to the given plane, so this vector \vec{n} will be normal to the required plane also.

The equation of plane passing through (x_1, y_1, z_1) and having normal \vec{n} is:

$$(\vec{r} - \vec{a}) \cdot \vec{n} = 0$$

$$\text{i.e., } \{(x - x_1)\vec{i} + (y - y_1)\vec{j} + (z - z_1)\vec{k}\} \cdot \vec{n} = 0$$

$$\text{or, } \{(x - 1)\vec{i} + (y + 1)\vec{j} + (z - 3)\vec{k}\} \cdot (3\vec{i} + \vec{j} + \vec{k}) = 0 \quad [\because (x_1, y_1, z_1) = (1, -1, 3)]$$

$$\text{or, } (x - 1) \times 1 + (y + 1) \times (-1) + (z - 3) \times 3 = 0$$

$$\text{or, } 3x + y + z - 5 = 0$$

Plane through Three Given Points

Let O be the origin and let the plane pass through the points A, B, and C whose position vectors are \vec{a} , \vec{b} , and \vec{c} respectively. Let $P(x, y, z)$ be any point on the plane whose position vector is \vec{r} . Then the points P, A, B, C all lie on the same plane. Now, $\vec{AB} \times \vec{AC}$ is a vector which is perpendicular to the plane and \vec{AP} lies on the plane. So,

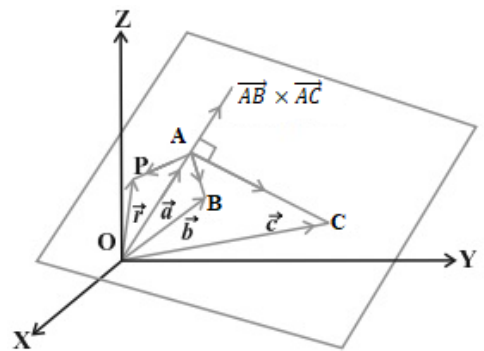
$$\vec{AP} \cdot (\vec{AB} \times \vec{AC}) = 0$$

$$\text{i.e., } (\vec{r} - \vec{a}) \cdot \vec{n} = 0 \quad \text{where } \vec{n} = \vec{AB} \times \vec{AC}$$

Since $\vec{n} = \vec{AB} \times \vec{AC} = (\vec{b} - \vec{a}) \times (\vec{c} - \vec{a})$, so the above equation becomes

$$(\vec{r} - \vec{a}) \cdot [(\vec{b} - \vec{a}) \times (\vec{c} - \vec{a})] = 0$$

Which is the required equation of the plane.



Illustration

Find the equation of the plane passing through the points (2, 4, 5), (1, 5, 7), and (-1, 6, 8).

Solution

Let the points be A(2, 4, 5), B(1, 5, 7), and C(-1, 6, 8). Let \vec{OA} , \vec{OB} , \vec{OC} be the position vectors of the points A, B, C so that $\vec{OA} = 2\vec{i} + 4\vec{j} + 5\vec{k}$, $\vec{OB} = \vec{i} + 5\vec{j} + 7\vec{k}$, and $\vec{OC} = -\vec{i} + 6\vec{j} + 8\vec{k}$.

$$\text{Now, } \vec{AB} = \vec{OB} - \vec{OA} = \vec{i} + 5\vec{j} + 7\vec{k} - (2\vec{i} + 4\vec{j} + 5\vec{k}) = -\vec{i} + \vec{j} + 2\vec{k}$$

$$\vec{AC} = \vec{OC} - \vec{OA} = -\vec{i} + 6\vec{j} + 8\vec{k} - (2\vec{i} + 4\vec{j} + 5\vec{k}) = -3\vec{i} + 2\vec{j} + 3\vec{k}$$

Since \vec{AB} and \vec{AC} lie on the same plane, so $\vec{AB} \times \vec{AC}$ is normal to the plane.

$$\vec{n} = \vec{AB} \times \vec{AC} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -1 & 1 & 2 \\ -3 & 2 & 3 \end{vmatrix} = -\vec{i} - 3\vec{j} + \vec{k}$$

Let P(x, y, z) be any point on the plane, such that, $\vec{OP} = x\vec{i} + y\vec{j} + z\vec{k}$.

$$\text{Now, } \vec{AP} = \vec{OP} - \vec{OA} = (x - 2)\vec{i} + (y - 4)\vec{j} + (z - 5)\vec{k}$$

\vec{AP} lies on a plane and \vec{n} is normal to the plane, thus, the required equation of the plane is

$$\vec{AP} \cdot \vec{n} = 0$$

$$\text{i.e., } [(x - 2)\vec{i} + (y - 4)\vec{j} + (z - 5)\vec{k}] \cdot (-\vec{i} - 3\vec{j} + \vec{k}) = 0$$

$$\text{or, } -x - 3y + z + 9 = 0$$

Which is the required equation of the plane.

Illustration

Find the equation of the plane passing through the point (2, 4, 5) and perpendicular to the line

$$\frac{x-5}{1} = \frac{y-1}{3} = \frac{z}{4}.$$

Solution

Here,

$$\frac{x-5}{1} = \frac{y-1}{3} = \frac{z}{4} = \lambda$$

$$\Rightarrow x - 5 = \lambda, y - 1 = 3\lambda, z = 4\lambda$$

$$\Rightarrow (x - 5)\vec{i} + (y - 1)\vec{j} + z\vec{k} = \lambda(\vec{i} + 3\vec{j} + 4\vec{k})$$

$$\Rightarrow (x - 5)\vec{i} + (y - 1)\vec{j} + z\vec{k} = \lambda\vec{n}$$

This line is parallel to the vector $\vec{n} = \vec{i} + 3\vec{j} + 4\vec{k}$.

Now, the equation of plane through (2, 4, 5) and perpendicular to \vec{n} is $(\vec{r} - \vec{a}) \cdot \vec{n} = 0$

$$\text{i.e., } \{ (x\vec{i} + y\vec{j} + z\vec{k}) - (2\vec{i} + 4\vec{j} + 5\vec{k}) \} \cdot (\vec{i} + 3\vec{j} + 4\vec{k}) = 0$$

$$\text{or, } \{ (x - 2)\vec{i} + (y - 4)\vec{j} + (z - 5)\vec{k} \} \cdot (\vec{i} + 3\vec{j} + 4\vec{k}) = 0$$

$$\text{or, } x + 3y + 4z = 34$$

Which is the required equation of the plane.

Exercise for Reader

1. Find the equation of the plane passing through the point $A(0, 2, -1)$ and normal to the vector $\vec{n} = 3\vec{i} - 2\vec{j} - \vec{k}$.
2. Find the equation of the plane passing through the point $(2, -3, 5)$ and parallel to the plane $x + 2y - z = 5$.
3. Find the equation of the plane passing through the points $(0, 0, 1)$, $(2, 0, 0)$, and $(0, 3, 0)$.
4. Find the equation of the plane passing through the points $(4, 1, 3)$ and perpendicular to the line $\frac{x-1}{3} = \frac{y-2}{4} = \frac{z}{5}$.