Lecture 30

Learning Objectives

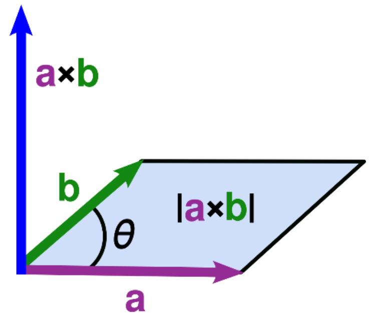
At the end of this class, students should be able to:

- understand the concept of vector product of two vectors
- identify the equation of line passing through a point and parallel to the given vector
- identify the equation of line passing through two points
- solve related problems

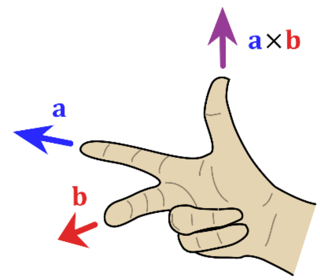
Vector Product (or Cross Product) of two Vectors

The vector product of any two vectors $\vec{a}$ and $\vec{b}$ is denoted by $\vec{a}\times\vec{b}$ and defined as $\vec{a}\times\vec{b} = \hat{n}|\vec{a}||\vec{b}|\sin\theta$. Where $\hat{n}$ is the unit vector perpendicular to the plain containing $\vec{a}$ and $\vec{b}$ and the direction of $\hat{n}$ is taken according to right handed screw system.

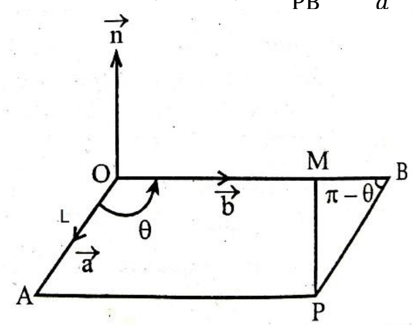
Vector Product of two Vectors



Right-hand Rule



Geometrical Interpretation: In the following figure, let $\overrightarrow{OA} = \vec{a}$ and $\overrightarrow{OB} = \vec{b}$ and let θ be the angle between $\vec{a}$ and $\vec{b}$. Complete the parallelogram OAPB with OA and OB as adjacent sides. From P draw perpendicular to OB. Then $\sin(\pi-\theta) = \frac{\text{PM}}{\text{PB}} = \frac{\text{PM}}{a} \Rightarrow \text{PM} = a\sin(\pi-\theta) = a\sin\theta$.



If A is the area of parallelogram, then A = OB×PM = $ab\sin\theta$

Also, $|\vec{a}\times\vec{b}| = ab\sin\theta$

$\therefore$ $|\vec{a}\times\vec{b}|$ = Area of parallelogram OAPB

Thus, $\vec{a} \times \vec{b}$ is vector whose magnitude is equal to the area of the parallelogram formed with $\vec{a}$ and $\vec{b}$ as adjacent sides.

Properties of Vector Product

1. If $\vec{a}$ and $\vec{b}$ are parallel, i.e., $\theta = 0^o$ or $\theta = \pi$, then $\vec{a} \times \vec{b} = 0$.

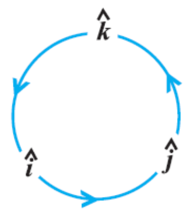
   In particular, $\vec{a} \times \vec{a} = 0$; $\vec{b} \times \vec{b} = 0$; $\vec{i} \times \vec{i} = 0$; $\vec{j} \times \vec{j} = 0$; $\vec{k} \times \vec{k} = 0$.

2. If $\vec{a}$ and $\vec{b}$ are perpendicular, then $\vec{a} \times \vec{b} = ab\hat{n}$.

   In particular,

   i) $\vec{i} \times \vec{j} = \vec{k}$; $\vec{j} \times \vec{k} = \vec{i}$; $\vec{k} \times \vec{i} = \vec{j}$

   ii) $\vec{j} \times \vec{i} = -\vec{k}$; $\vec{i} \times \vec{k} = -\vec{j}$; $\vec{k} \times \vec{j} = -\vec{i}$



3. The vector product is not commutative, i.e., $\vec{a} \times \vec{b} \neq \vec{b} \times \vec{a}$
4. The vector product is distributive, i.e., $\vec{a} \times (\vec{b} + \vec{c}) = \vec{a} \times \vec{b} + \vec{a} \times \vec{c}$

Illustration

If $\vec{a} = 4\vec{i} + 3\vec{j} + \vec{k}$, $\vec{b} = 2\vec{i} - \vec{j} + 2\vec{k}$, find the unit vector $\hat{n}$ perpendicular to vectors $\vec{a}$ and $\vec{b}$ such that $\vec{a}, \vec{b}, \hat{n}$ form a right-handed system. Also find the angle between $\vec{a}$ and $\vec{b}$.

Solution

Here, $\vec{a} = 4\vec{i} + 3\vec{j} + \vec{k}$, $\vec{b} = 2\vec{i} - \vec{j} + 2\vec{k}$ then $|\vec{a}| = \sqrt{26}$, $|\vec{b}| = 3$

Now,

$$\vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 4 & 3 & 1 \\ 2 & -1 & 2 \end{vmatrix} = 7\vec{i} - 6\vec{j} - 10\vec{k}$$

$$|\vec{a} \times \vec{b}| = \sqrt{185}$$

$$\hat{n} = \frac{\vec{a} \times \vec{b}}{|\vec{a} \times \vec{b}|} = \frac{1}{\sqrt{185}}(7\vec{i} - 6\vec{j} - 10\vec{k})$$

We know that $|\vec{a} \times \vec{b}| = |\vec{a}||\vec{b}| \sin\theta$

or, $\quad \sin\theta = \frac{|\vec{a} \times \vec{b}|}{|\vec{a}||\vec{b}|} = \frac{\sqrt{185}}{\sqrt{26} \times 3}$

$\therefore \quad \theta = \sin^{-1}\left(\frac{\sqrt{185}}{3\sqrt{26}}\right)$

Illustration

If $\vec{a}.\vec{b} = \vec{a}.\vec{c}$ and $\vec{a} \times \vec{b} = \vec{a} \times \vec{c}$ where $\vec{a}$ is a proper vector (i.e., $\vec{a} \neq \vec{0}$), show that $\vec{b} = \vec{c}$.

Solution

We have $\quad \vec{a}.\vec{b} = \vec{a}.\vec{c} \Rightarrow \vec{a}.(\vec{b} - \vec{c}) = 0 \quad$ (i)

$$\vec{a} \times \vec{b} = \vec{a} \times \vec{c} \Rightarrow \vec{a} \times (\vec{b} - \vec{c}) = 0 \qquad \text{(ii)}$$

Equation (i) states that $\vec{b} - \vec{c}$ is perpendicular to $\vec{a}$.

Equation (ii) states that $\vec{b} - \vec{c}$ is parallel to $\vec{a}$.

This is possible only when $\vec{b} - \vec{c} = 0 \Rightarrow \vec{b} = \vec{c}$

Which is required.

Illustration

Find the area of the triangle whose vertices are A = (1, 2, 2), B = (3, 4, 5) and C = (5, 6, 4).

Solution

Let O be the origin, then $\overrightarrow{OA} = \vec{\imath} + 2\vec{\jmath} + 2\vec{k}$, $\overrightarrow{OB} = 3\vec{\imath} + 4\vec{\jmath} + 5\vec{k}$, and $\overrightarrow{OC} = 5\vec{\imath} + 6\vec{\jmath} + 4\vec{k}$
Now, $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = 3\vec{\imath} + 4\vec{\jmath} + 5\vec{k} - (\vec{\imath} + 2\vec{\jmath} + 2\vec{k}) = 2\vec{\imath} + 2\vec{\jmath} + 3\vec{k}$ and

$$\overrightarrow{AC} = \overrightarrow{OC} - \overrightarrow{OA} = 5\vec{\imath} + 6\vec{\jmath} + 4\vec{k} - (\vec{\imath} + 2\vec{\jmath} + 2\vec{k}) = 4\vec{\imath} + 4\vec{\jmath} + 2\vec{k}$$

Now, the area of the triangle whose adjacent sides are AB and AC is given by $\frac{1}{2}\left|\overrightarrow{AB} \times \overrightarrow{AC}\right|$

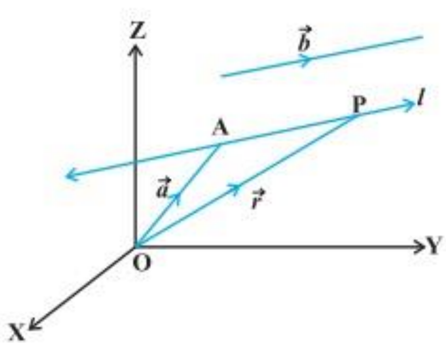
Here, $\overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ 2 & 2 & 3 \\ 4 & 4 & 2 \end{vmatrix} = -8\vec{\imath} + 8\vec{\jmath} + 0\vec{k}$

Thus, $\left|\overrightarrow{AB} \times \overrightarrow{AC}\right| = \sqrt{128} = 8\sqrt{2}$

Hence, the required area of a triangle = $\frac{1}{2}\left|\overrightarrow{AB} \times \overrightarrow{AC}\right| = 4\sqrt{2}$

Equation of a Line through a Given Point A and parallel to a Given Vector $\vec{b}$

Let l be the line in the space passing through a given point A and parallel to a vector $\vec{b}$. Let O be the origin and P be any point on the line whose position vector is $\vec{r}$, i.e., $\vec{r} = \overrightarrow{OP}$. Also, $\overrightarrow{OA} = \vec{a}$. Since $\overrightarrow{AP}$ is parallel to vector $\vec{b}$, so $\overrightarrow{AP} = \lambda\vec{b}$ where λ is a scalar.



Now, $\overrightarrow{OP} = \overrightarrow{OA} + \overrightarrow{AP}$

$$\vec{r} = \vec{a} + \lambda\vec{b} \qquad \text{(i)}$$

Since P is any point on the line, so the above equation is a vector equation of a line.

If the line (i) passes through the origin, then equation (i) reduces to

$$\vec{r} = \lambda\vec{b}$$

Illustration

Find the equation of line through (2, -9, 5) and parallel to $2\vec{i} + 5\vec{j} + 6\vec{k}$.

Solution

Equation of a line passing through a point A and parallel to vector $\vec{b}$ is given by

$$\vec{r} = \vec{a} + \lambda\vec{b}$$

Here, $\vec{a} = 2\vec{i} - 9\vec{j} + 5\vec{k}$ and $\vec{b} = 2\vec{i} + 5\vec{j} + 6\vec{k}$

Let $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$

Thus, above equation becomes

$$x\vec{i} + y\vec{j} + z\vec{k} = 2\vec{i} - 9\vec{j} + 5\vec{k} + \lambda(2\vec{i} + 5\vec{j} + 6\vec{k})$$

or, $\quad (x-2)\vec{i} + (y+9)\vec{j} + (z-5)\vec{k} = \lambda(2\vec{i} + 5\vec{j} + 6\vec{k})$

Equating the coefficients of $\vec{i}$, $\vec{j}$ and $\vec{k}$ on both sides, we get

$$x - 2 = 2\lambda,\ y + 9 = 5\lambda,\ z - 5 = 6\lambda$$

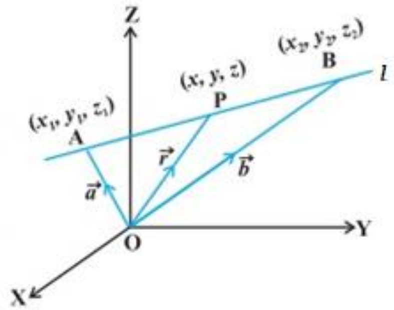
$\Rightarrow \quad \frac{x-2}{2} = \lambda,\ \frac{y+9}{5} = \lambda,\ \frac{z-5}{6} = \lambda$

$\Rightarrow \quad \frac{x-2}{2} = \frac{y+9}{5} = \frac{z-5}{6}$

Which is the required

Equation of a Line through two Points A and B

Let O be the origin and let the line l passes through the points A and B whose position vectors are $\overrightarrow{OA} = \vec{a}$ and $\overrightarrow{OB} = \vec{b}$. Let P be any point on the line whose position vector is $\vec{r}$, i.e., $\vec{r} = \overrightarrow{OP}$.



Here, $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = \vec{b} - \vec{a}$

Since $\overrightarrow{AP}$ is collinear with $\overrightarrow{AB}$, so $\overrightarrow{AP} = \lambda\overrightarrow{AB} = \lambda(\vec{b} - \vec{a})$

Now, $\overrightarrow{OP} = \overrightarrow{OA} + \overrightarrow{AP}$

$$\vec{r} = \vec{a} + \lambda(\vec{b} - \vec{a})$$

This is the vector equation of the line passing through two points.

Illustration

Find the equation of line passing through the points A(-3, 2, -3) and B(1, -1, 4).

Solution

The equation of line passing through two points A and B is:

$$\vec{r} = \vec{a} + \lambda(\vec{b} - \vec{a})$$

Here, $\vec{a} = -3\vec{i} + 2\vec{j} - 3\vec{k}$ and $\vec{b} = \vec{i} - \vec{j} + 4\vec{k}$

Let $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$

Then, above equation becomes

$$x\vec{i} + y\vec{j} + z\vec{k} = -3\vec{i} + 2\vec{j} - 3\vec{k} + \lambda\{(\vec{i} - \vec{j} + 4\vec{k}) - (-3\vec{i} + 2\vec{j} - 3\vec{k})\}$$

or, $\quad (x+3)\vec{i} + (y-2)\vec{j} + (z+3)\vec{k} = \lambda(4\vec{i} - 3\vec{j} + 7\vec{k})$

Equating the coefficients of $\vec{i}$, $\vec{j}$ and $\vec{k}$ on both sides, we get

$$x + 3 = 4\lambda,\ y - 2 = -3\lambda,\ z + 3 = 7\lambda$$

$\Rightarrow \quad \frac{x+3}{4} = \lambda,\ \frac{y-2}{-3} = \lambda,\ \frac{z+3}{7} = \lambda$

$\Rightarrow \quad \frac{x+3}{4} = \frac{y-2}{-3} = \frac{z+3}{7}$

Which is the required

Exercise for Reader

1. Find the unit vector $\hat{n}$ normal to the plane containing $\vec{a} = 3\vec{i} - 2\vec{j} + 4\vec{k}$ and $\vec{b} = \vec{i} + \vec{j} - 2\vec{k}$.
2. Find the area of the triangle whose vertices are A = (1, -1, 2), B = (2, 0, 1) and C = (0, 2, 1).
3. Prove tat $\vec{a} \times (\vec{b} + \vec{c}) + \vec{b} \times (\vec{c} + \vec{a}) + \vec{c} \times (\vec{a} + \vec{b}) = 0$
4. Find the equation of line through (3, -4, -1) and parallel to $\vec{i} + \vec{j} + \vec{k}$.
5. Find the equation of line passing through the points A(1, 2, 0) and B(1, 1, -1).