

## Lecture 29

### Learning Objectives

At the end of this class, students should be able to:

- understand the notion of vector and scalar.
- perform basic operations: addition, subtraction, scalar multiplication, etc.
- find scalar product of two vectors
- solve related problems

### Scalars and Vectors

A quantity which is completely specified by a certain number associated with a suitable unit without any mention of direction in space is known as scalar. For example, time, mass, length, volume, density, temperature, energy, distance, speed etc. The number describing the quantity of a particular scalar is known as its magnitude. The scalars are added subtracted, multiplied and divided by the usual arithmetical laws.

There are quantities which require both magnitude and direction for their complete description. Such quantities are called vectors. For example, if a particle is displaced from a position  $A$  to a position  $B$ , then the displacement  $AB$  is a vector. Its magnitude is the length  $AB$  and its direction is from  $A$  to  $B$ . It is thus a directed line segment. A directed line segment is called a vector.

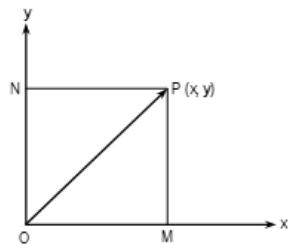
### Vector Representation

A vector quantity is represented by a straight-line segment. It is generally denoted by a lower-case letter or a combination of two letters with an arrow over it or a bold face letter. The displacement vector in above example may therefore; be written as:  $\vec{a}$ , or  $\mathbf{a}$ , or  $\overrightarrow{AB}$ , or  $\mathbf{AB}$ . The arrow head indicate the direction from  $A$  to  $B$ . Here point  $A$  is called initial point and point  $B$  is called the terminal point.



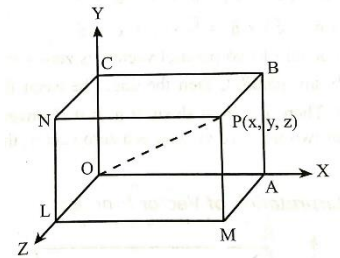
### Representation of Vectors in Terms of Coordinates

Let  $P(x, y)$  be any point on the plane. Then  $OM = x$  and  $ON = y$ . Here, the  $\overrightarrow{OP}$  is the directed line segment  $OP$ . This vector may be uniquely identified with the point  $P(x, y)$  so,  $\overrightarrow{OP} = (x, y)$ . Since the vector is in the plane, therefore it is called a plane vector.



A vector in 3-dimensional space is given by  $\overrightarrow{OP} = (x, y, z)$ . Since  $O$  is the origin of the coordinate system, then the  $\overrightarrow{OP}$  is also called the position vector of  $P$ .

In general, an ordered set of  $n$  numbers  $\vec{a} = (a_1, a_2, a_3, \dots, a_n)$  is called a vector of order  $n$ . It is a vector in  $n$ -dimensional space. The elements  $a_1, a_2, a_3, \dots, a_n$  are called the components of  $\vec{a}$ .



## Modulus of a Vector

Let  $\vec{a} = (a_1, a_2, a_3)$  be a vector of order 3. Then the modulus, or length, or magnitude of a vector  $\vec{a}$  is denoted by  $|\vec{a}|$  or simply  $a$ . It is defined by

$$|\vec{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2}$$

## Types of Vectors

1. **Null Vector or Zero Vector:** A vector whose magnitude is zero is called a null vector. It is generally denoted by  $O$ . The initial and terminating point of a null vector are coincident. For example,  $(0, 0)$  and  $(0,0,0)$  are null vectors in 2-dimensional space and 3-dimensional space respectively.

2. **Unit Vector:** A vector whose magnitude is unity is called a unit vector. It is denoted by  $\hat{a}$  and defined as  $\hat{a} = \frac{\vec{a}}{|\vec{a}|}$ . If  $\vec{a} = (a_1, a_2)$ , then  $\hat{a} = \frac{\vec{a}}{|\vec{a}|} = \frac{(a_1, a_2)}{\sqrt{a_1^2 + a_2^2}}$  is a unit vector in the direction of  $\vec{a}$ .

The unit vectors in 3-dimensional space along the positive directions of coordinate axes  $OX$ ,  $OY$ , and  $OZ$  are  $(1, 0, 0)$ ,  $(0, 1, 0)$  and  $(0, 0, 1)$ . These unit vectors are also denoted by  $\vec{i}$ ,  $\vec{j}$ , and  $\vec{k}$ . Thus,

$$\vec{i} = (1, 0, 0), \vec{j} = (0, 1, 0), \text{ and } \vec{k} = (0, 0, 1).$$

3. **Equal Vectors:** Let  $\vec{a} = (a_1, a_2, a_3)$  and  $\vec{b} = (b_1, b_2, b_3)$  be two vectors of order 3. Then  $\vec{a} = \vec{b}$  if and only if  $a_1 = b_1$ ,  $a_2 = b_2$ , and  $a_3 = b_3$ .

4. **Parallel Vectors, like and unlike vectors:** Let  $\vec{a}$  and  $\vec{b}$  be two vectors. If they lie along the same or parallel lines, irrespective of their sense, they are called parallel vectors. However, if their sense is also the same, they are called like vectors and if their sense is opposite, they are called unlike. Mathematically, if  $\vec{a} = k\vec{b}$ , where  $k$  is a positive number, then  $\vec{a}$  and  $\vec{b}$  are called like vectors. If  $k$  is negative, they are said to be unlike vectors.

5. **Negative vector:** Let  $\vec{AB}$  be a given vector. Then the negative of the given vector  $\vec{AB}$  is  $-\vec{AB}$  or  $\vec{BA}$ . It has the same magnitude but the direction is opposite.



## Sum of two Vectors

Let  $\vec{u}$  and  $\vec{v}$  are two vectors. The sum of  $\vec{u}$  and  $\vec{v}$  is denoted by  $\vec{u} + \vec{v}$  is a vector.

If a particle is displaced from  $A$  to  $B$ , we denote the displacement by  $\vec{AB}$ . Let a particle be given two displacements: one from  $A$  to  $B$  and another from  $B$  to  $C$ , then the particle is finally displaced from  $A$  to  $C$ .

This suggests that

$$\vec{AB} + \vec{BC} = \vec{AC}$$

Thus,  $\vec{AC}$  is the sum of  $\vec{AB}$  and  $\vec{BC}$ .

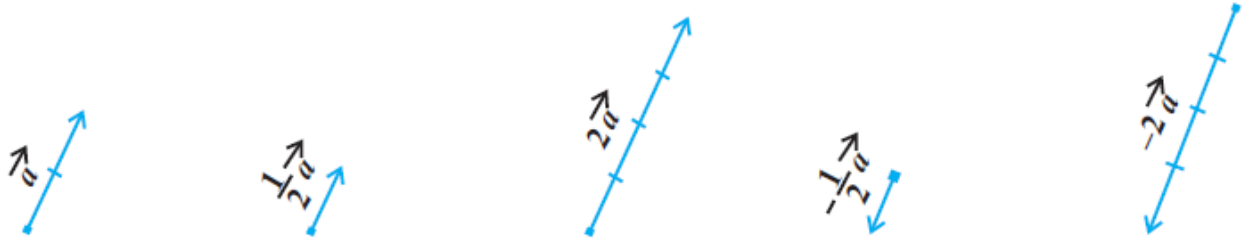


### Scalar Multiplication

Let  $\vec{a} = (a_1, a_2, a_3)$  be a vector and  $k$  be a scalar. Then the product  $k\vec{a}$  is known as scalar multiplication and defined by

$$k\vec{a} = k(a_1, a_2, a_3) = (ka_1, ka_2, ka_3)$$

A geometric visualization of multiplication of a vector by a scalar is given in the following figures.

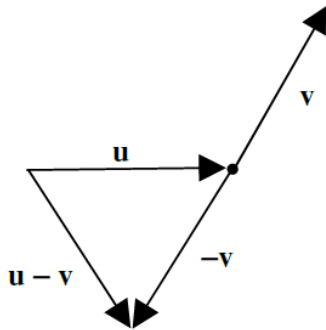


### Difference of two Vector

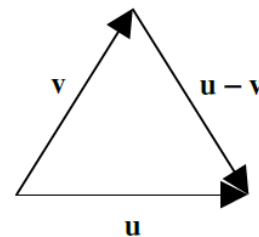
Let  $\vec{u}$  and  $\vec{v}$  are two vectors. The difference of  $\vec{u}$  and  $\vec{v}$  is denoted by  $\vec{u} - \vec{v}$  (or  $\vec{u} + (-\vec{v})$ ). To construct the vector  $\vec{u} - \vec{v}$ , we can either

- i) Construct the sum of  $\vec{u}$  and  $-\vec{v}$ ; or
- ii) Position  $\vec{u}$  and  $\vec{v}$  so that their initial points (tails) coincide, then the vector from the terminal point (head) of  $\vec{v}$  to the terminal point of  $\vec{u}$  is vector  $\vec{u} - \vec{v}$ .

(i)



(ii)



### Position Vector

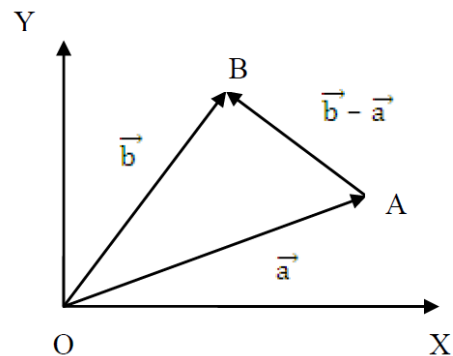
A vector that starts from the origin (O) is called a position vector. In the following diagram, point A has the position vector  $\vec{a}$  and point B has the position vector  $\vec{b}$ .

We know that,

$$\vec{OA} + \vec{AB} = \vec{OB}$$

or,  $\vec{AB} = \vec{OB} - \vec{OA}$

or,  $\vec{AB} = \vec{b} - \vec{a}$



Thus,  $\overrightarrow{AB}$  = position vector of point B – position vector of point A

### Vectors in terms of $\vec{i}, \vec{j}, \vec{k}$

Let P(x, y, z) be any point in the space. Then its position vector is  $\overrightarrow{OP}$ . Let  $\vec{r} = \overrightarrow{OP} = (x, y, z)$ . Then

$$\vec{r} = x(1, 0, 0) + y(0, 1, 0) + z(0, 0, 1) = x\vec{i} + y\vec{j} + z\vec{k}$$

Thus,  $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$

Here x, y, z are called the components of  $\vec{r}$  and the modulus of  $\vec{r}$  is given by

$$|\vec{r}| = \sqrt{x^2 + y^2 + z^2}$$

### Dot Product (or Scalar Product) of two Vectors

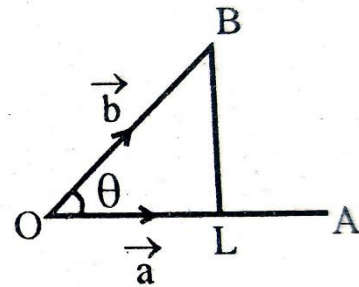
The dot product is a method for multiplying two vectors. The dot product of any two vectors  $\vec{a}$  and  $\vec{b}$  is denoted by  $\vec{a} \cdot \vec{b}$  and defined as  $\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}| \cos \theta = ab \cos \theta$ . Because this product of two vectors is a scalar, the dot product is sometimes referred to as the scalar product. The dot product is also used to find an angle between two vectors. Here,  $\cos \theta = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}||\vec{b}|}$

**Geometrical Interpretation:** Let  $\overrightarrow{OA} = \vec{a}$  and  $\overrightarrow{OB} = \vec{b}$  and let  $\theta$  be the angle between  $\vec{a}$  and  $\vec{b}$ . Draw BL perpendicular to OA.

Then

$$\begin{aligned} \vec{a} \cdot \vec{b} &= ab \cos \theta \\ &= a(b \cos \theta) \\ &= (OA)(OL) \\ &= |\vec{a}| \times (\text{Projection of } \vec{b} \text{ on } \vec{a}) \end{aligned}$$

Thus, Projection of  $\vec{b}$  on  $\vec{a} = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}|} = \vec{b} \cdot \hat{a}$



This is the scalar projection of  $\vec{b}$  on  $\vec{a}$ . The scalar projection of  $\vec{b}$  on  $\vec{a}$  is also called the scalar component of  $\vec{b}$  in the direction of  $\vec{a}$ .

Thus,  $\vec{a} \cdot \vec{b} = |\vec{a}| \times (\text{scalar component of } \vec{b} \text{ in the direction of } \vec{a})$

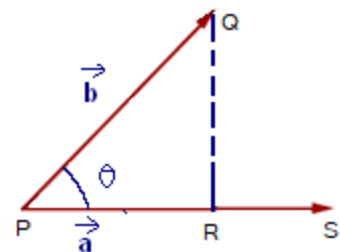
or,  $\vec{a} \cdot \vec{b} = |\vec{b}| \times (\text{scalar component of } \vec{a} \text{ in the direction of } \vec{b})$

**Vector Projection:** Vector projection of  $\vec{b}$  onto  $\vec{a}$  is a vector  $\overrightarrow{PR}$ .

Its magnitude is  $|\vec{b}| \cos \theta$  and direction is  $\frac{\vec{a}}{|\vec{a}|}$ .

Thus, vector projection of  $\vec{b}$  onto  $\vec{a}$  is  $(|\vec{b}| \cos \theta) \frac{\vec{a}}{|\vec{a}|}$

$$\begin{aligned} &= |\vec{b}| \times \frac{\vec{a} \cdot \vec{b}}{|\vec{a}||\vec{b}|} \times \frac{\vec{a}}{|\vec{a}|} \\ &= \left( \frac{\vec{a} \cdot \vec{b}}{|\vec{a}||\vec{a}|} \right) \vec{a} \end{aligned}$$



$$= \left( \frac{\vec{a} \cdot \vec{b}}{\vec{a} \cdot \vec{a}} \right) \vec{a}$$

Similarly, vector projection of  $\vec{a}$  onto  $\vec{b}$  is  $\left( \frac{\vec{a} \cdot \vec{b}}{\vec{b} \cdot \vec{b}} \right) \vec{b}$

Scalar projection is a scalar quantity whereas vector projection is a vector quantity.

### Properties of Dot Product

1. If  $\theta = 90^\circ$ , i.e., the vectors  $\vec{a}$  and  $\vec{b}$  are perpendicular (or orthogonal), then  $\vec{a} \cdot \vec{b} = 0$ . Since the vectors:  $\vec{i}, \vec{j}$ , and  $\vec{k}$  are perpendicular to each other, so,  $\vec{i} \cdot \vec{j} = 0$ ,  $\vec{j} \cdot \vec{k} = 0$ ,  $\vec{k} \cdot \vec{i} = 0$ .
2. If  $\vec{a}$  is parallel to  $\vec{b}$ , then either  $\theta = 0^\circ$  or  $\theta = \pi$ , and  $\vec{a} \cdot \vec{b} = |\vec{a}| |\vec{b}|$  or  $-|\vec{a}| |\vec{b}|$ .  
Also  $\vec{i} \cdot \vec{i} = 1$ ,  $\vec{j} \cdot \vec{j} = 1$ ,  $\vec{k} \cdot \vec{k} = 1$ .
3.  $\vec{a} \cdot \vec{b} = +ve$  or  $-ve$  or zero as the angle between the vectors is acute, obtuse or right angle.
4.  $\vec{a} \cdot \vec{a} > 0$  if  $a \neq 0$ .
5.  $\vec{a} \cdot \vec{a} = |\vec{a}| |\vec{a}| \cos 0 = |\vec{a}|^2 = a^2$
6. The dot product is not commutative, i.e.,  $\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a}$
7. The dot product is distributive, i.e.,  $\vec{a} \cdot (\vec{b} + \vec{c}) = \vec{a} \cdot \vec{b} + \vec{a} \cdot \vec{c}$

#### Illustration

If  $\vec{a} = 2\vec{i} + \vec{j} - 3\vec{k}$ ,  $\vec{b} = 3\vec{i} - 2\vec{j} - \vec{k}$ , find  $\vec{a} \cdot \vec{b}$ . Also find the angle between  $\vec{a}$  and  $\vec{b}$ .

*Solution*

Here,  $\vec{a} = 2\vec{i} + \vec{j} - 3\vec{k}$ ,  $\vec{b} = 3\vec{i} - 2\vec{j} - \vec{k}$  then  $|\vec{a}| = \sqrt{14}$ ,  $|\vec{b}| = \sqrt{14}$

Now,  $\vec{a} \cdot \vec{b} = (2\vec{i} + \vec{j} - 3\vec{k}) \cdot (3\vec{i} - 2\vec{j} - \vec{k}) = 2 \times 3 + 1 \times (-2) + (-3) \times (-1) = 7$

We know that  $\cos \theta = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}| |\vec{b}|} = \frac{7}{\sqrt{14} \times \sqrt{14}} = \frac{1}{2}$

$$\theta = \cos^{-1} \left( \frac{1}{2} \right) = 60^\circ$$

#### Illustration

If  $\vec{a} = 2\vec{i} - 4\vec{j} + \sqrt{5}\vec{k}$ ,  $\vec{b} = -2\vec{i} + 4\vec{j} - \sqrt{5}\vec{k}$ , find

- i) the scalar component of  $\vec{b}$  in the direction of  $\vec{a}$ .
- ii) the vector projection of  $\vec{b}$  onto  $\vec{a}$ .

*Solution*

- i) We know that, the scalar component of  $\vec{b}$  in the direction of  $\vec{a}$  is given  $\frac{\vec{a} \cdot \vec{b}}{|\vec{a}|}$

$$= \frac{(2\vec{i} - 4\vec{j} + \sqrt{5}\vec{k}) \cdot (-2\vec{i} + 4\vec{j} - \sqrt{5}\vec{k})}{\sqrt{2^2 + (-4)^2 + (\sqrt{5})^2}}$$

$$\begin{aligned}
&= \frac{-4-16-5}{\sqrt{25}} \\
&= \frac{-25}{5} \\
&= -5
\end{aligned}$$

ii) the vector projection of  $\vec{b}$  onto  $\vec{a}$  is given by  $\left(\frac{\vec{a}\cdot\vec{b}}{\vec{a}\cdot\vec{a}}\right)\vec{a} = \left(\frac{\vec{a}\cdot\vec{b}}{|\vec{a}|^2}\right)\vec{a}$

$$\begin{aligned}
&= \frac{-25}{25}(2\vec{i} - 4\vec{j} + \sqrt{5}\vec{k}) \\
&= -2\vec{i} + 4\vec{j} - \sqrt{5}\vec{k}
\end{aligned}$$

### Exercise for Reader

1. Determine the value of  $p$  so that the vectors  $\vec{a} = 2\vec{i} + p\vec{j} + \vec{k}$ ,  $\vec{b} = 4\vec{i} - 2\vec{j} - 2\vec{k}$  are orthogonal.
2. Find the angle between the vectors  $\vec{a} = 2\vec{i} - 4\vec{j} + \sqrt{5}\vec{k}$ ,  $\vec{b} = -2\vec{i} + 4\vec{j} - \sqrt{5}\vec{k}$ .
3. If  $\vec{a} = 6\vec{i} + 3\vec{j} + 2\vec{k}$  and  $\vec{b} = \vec{i} - 2\vec{j} - 2\vec{k}$ , find the scalar component of  $\vec{a}$  in the direction of  $\vec{b}$ . Also, find the vector projection of  $\vec{a}$  onto  $\vec{b}$ .