Lecture 27

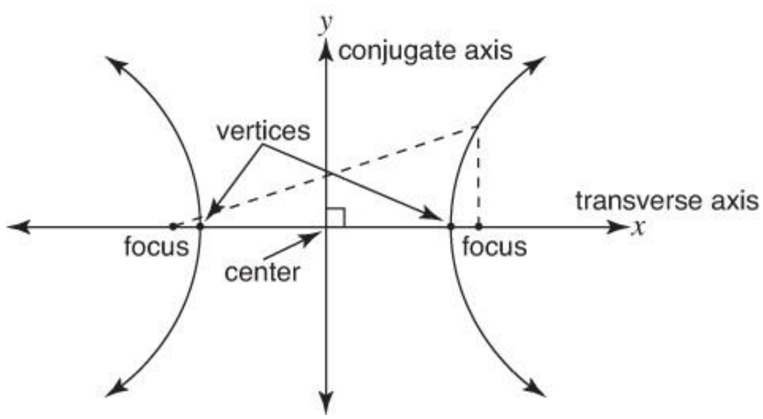
Learning Objectives

At the end of this class, students should be able to:

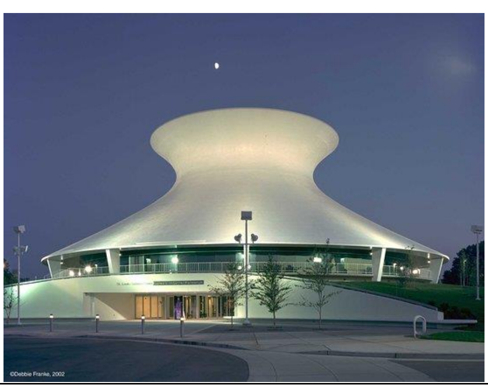
- derive the equation of hyperbola
- solve related problems

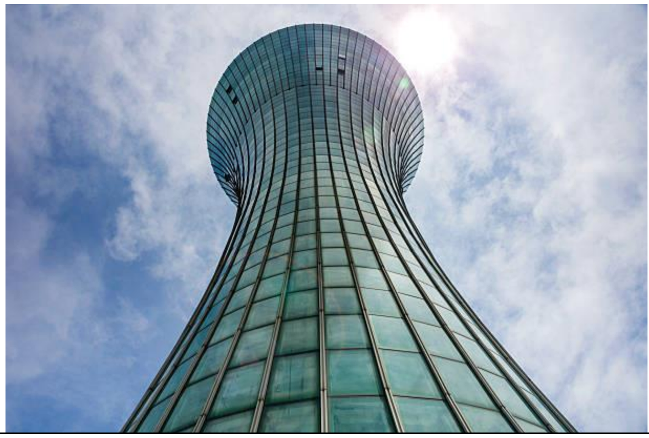
Hyperbola

A hyperbola is the set of all points in a plane such that the absolute value of the difference of the distance of each point from two fixed points is constant. Each fixed point is called a focus, and the point midway between the foci is called the center. The transverse axis is the line passing through the foci. The graph is made up of two parts called branches. Each branch intersects the transverse axis at a point called the vertex. The transverse axis length is the length of the line segment between the vertices. The other axis of symmetry through the center is the conjugate axis. A hyperbola has two asymptotes.



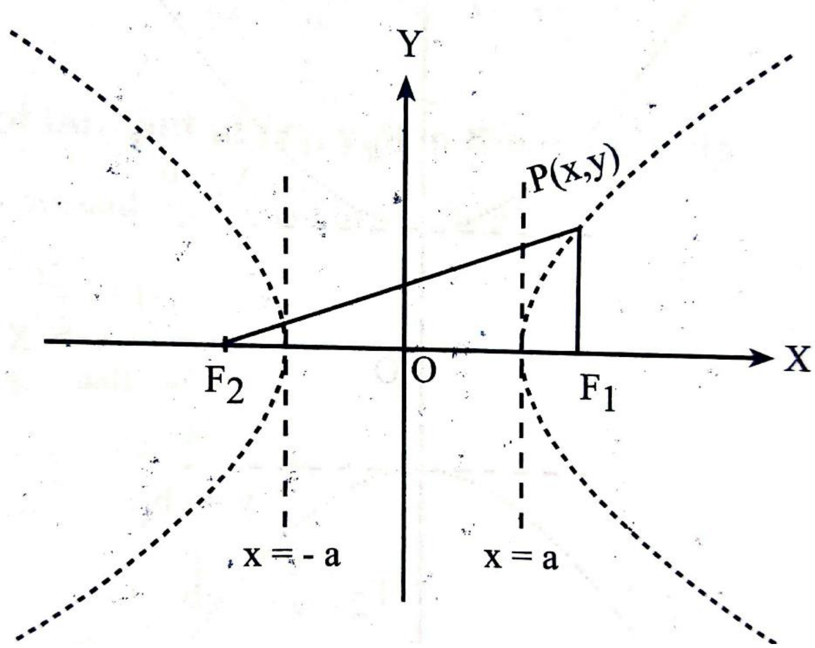
Hyperbola in Architecture





Standard Equation of Hyperbola

Let $F_1(c, 0)$ and $F_2(-c, 0)$ be foci of hyperbola and the constant difference is $2a$. let $P(x, y)$ be any point on the hyperbola. The difference of the distance $PF_2 - PF_1$ is denoted by $2a$.



i.e., $\quad PF_2 - PF_1 = 2a$

or, $\quad \sqrt{(x+c)^2 + y^2} - \sqrt{(x-c)^2 + y^2} = 2a$

or, $\quad \sqrt{(x+c)^2 + y^2} = 2a + \sqrt{(x-c)^2 + y^2}$

Squaring both sides, we get

$$(x+c)^2 + y^2 = \left(2a + \sqrt{(x-c)^2 + y^2}\right)^2$$

or, $\quad x^2 + 2cx + c^2 + y^2 = 4a^2 + x^2 - 2cx + c^2 + y^2 + 4a\sqrt{(x-c)^2 + y^2}$

or, $4a\sqrt{(x-c)^2+y^2} = 4(cx-a^2)$

Again, squaring both sides, we get

$$a^2(x^2-2cx+c^2+y^2) = a^4-2a^2cx+x^2c^2$$

or, $(a^2-c^2)x^2+a^2y^2 = a^2(a^2-c^2)$

or, $\frac{x^2}{a^2}+\frac{y^2}{(a^2-c^2)} = 1$ (i)

From figure $\Delta\, PF_2F_1$, we get

$$PF_2 - PF_1 < F_1F_2$$

i.e., $2a < 2c$

$\Rightarrow$ $a < c$

Thus, a^2-c^2 is negative and c^2-a^2 is positive, it has a real positive square root.

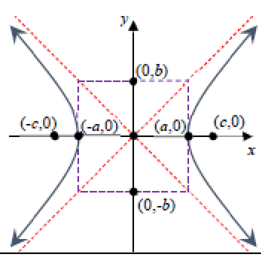
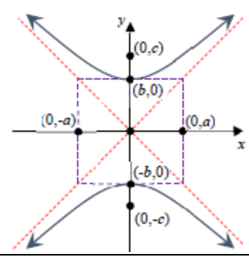
Put $c^2-a^2 = b^2 \Rightarrow b = \sqrt{c^2-a^2}$

Thus, equation (i) becomes

$$\frac{x^2}{a^2}-\frac{y^2}{b^2} = 1$$

Which is the equation of hyperbola having center (0, 0) and foci at $(\pm c, 0)$ where $b = \sqrt{c^2-a^2}$.

Equation of a Hyperbola Centered at the Origin in Standard Form

Opens	Horizontally	Vertically
Standard Equation	$\frac{x^2}{a^2}-\frac{y^2}{b^2} = 1$	$\frac{y^2}{b^2}-\frac{x^2}{a^2} = 1$
Graph		
Vertices	$(-a,0)$ and $(a,0)$	$(0,-b)$ and $(0,b)$
Foci	$(\pm c,0)$ where $b^2 = c^2-a^2$	$(0,\pm c)$ where $a^2 = c^2-b^2$
Asymptotes	$y = \pm\frac{b}{a}x$	$y = \pm\frac{b}{a}x$
Directrix	$x = \pm\frac{a}{e}$	$y = \pm\frac{b}{e}$
Length of Latus Rectum	$\frac{2b^2}{a}$	$\frac{2a^2}{b}$

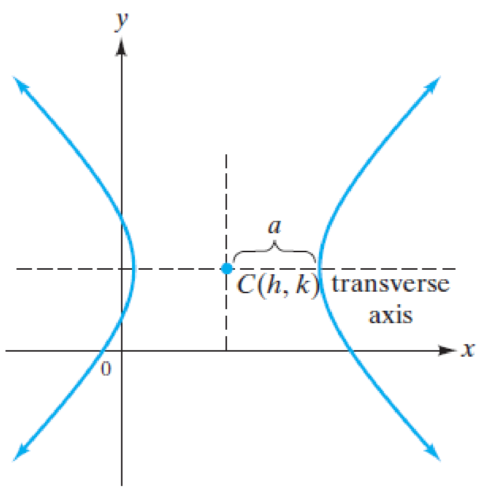
Transverse axis and its length	x-axis; $2a$	y-axis; $2b$
Conjugate axis and its length	y-axis; $2b$	x-axis; $2a$
Eccentricity	c/a	c/b

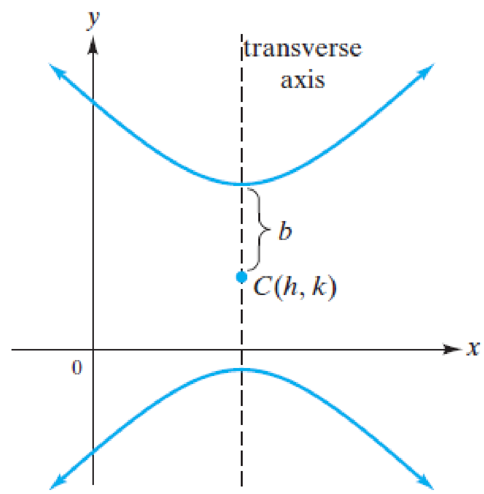
Shifting Hyperbola

The graph of a hyperbola can have its center at (h, k) rather than at the origin (0, 0). Horizontal and vertical translations are accomplished by replacing x with $x - h$ and y with $y - k$ in the standard form of the hyperbola's equation. Then the equation of hyperbola with its transverse axis parallel to x-axis and center at (h, k) is given by

$$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$$

Thus, the hyperbola $\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$ is generated if the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is shifted right h units and up k units as shown in the following figure. Similarly, the equation of hyperbola having center at (h, k) and transverse axis parallel to y-axis is $\frac{(y-k)^2}{b^2} - \frac{(x-h)^2}{a^2} = 1$.





Equation of a Hyperbola Centered at (*h, k*) in Standard Form

The standard form of an equation of a hyperbola centered at $C(h, k)$ depends on whether it opens horizontally or vertically. The table below gives the standard equation, vertices, foci, asymptotes for each.

Opens	Horizontally	Vertically
Standard Equation	$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$	$\frac{(y-k)^2}{b^2} - \frac{(x-h)^2}{a^2} = 1$
Vertices	$(h \pm a, k)$	$(h, k \pm b)$
Foci	$(h \pm c, k)$ where $b^2 = c^2 - a^2$	$(h, k \pm c)$ where $a^2 = c^2 - b^2$

Asymptotes	$y - k = \pm\frac{b}{a}(x - h)$	$y - k = \pm\frac{b}{a}(x - h)$
Directrix	$x = h \pm \frac{a}{e}$	$y = k \pm \frac{b}{e}$
Eccentricity	c/a	c/b

Illustration

Find the center, foci and vertices of the hyperbola $9x^2 - 16y^2 = 144$. Sketch the graph.

Solution

We have $\frac{x^2}{16} - \frac{y^2}{9} = 1$

The above equation is of the form $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$.

So, the center lies at (0, 0).

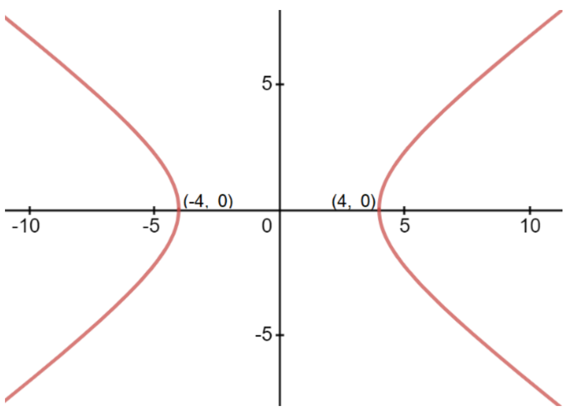
Here, $a = 4, b = 3$ then $c = \sqrt{a^2 + b^2} = \sqrt{4^2 + 3^2} = 5$

Since the y term is subtracted, the hyperbola opens horizontally.

Now, foci $(\pm c, 0)$ = $(\pm 5, 0)$ and

vertices $(\pm a, 0) = (\pm 4, 0)$

The graph of this hyperbola is as follows:



Illustration

Find the center, foci, vertices, eccentricity, length of transverse and conjugate axes, and directrix of the hyperbola $9x^2 - 16y^2 - 18x - 32y - 151 = 0$.

Solution

we have $9x^2 - 16y^2 - 18x - 32y - 151 = 0$.

or, $9(x^2 - 2x) - 16(y^2 + 2y) = 151$

or, $9(x^2 - 2x + 1) - 16(y^2 + 2y + 1) = 151 + 9 - 16$

or, $9(x-1)^2 - 16(y+1)^2 = 144$

or, $\frac{(x-1)^2}{4^2} - \frac{(y+1)^2}{3^2} = 1$

Now, comparing with $\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$, we get

$h = 1, k = -1, a = 4, b = 3$

and $c = \sqrt{a^2 + b^2} = \sqrt{4^2 + 3^2} = 5$

Since the y term is subtracted, the hyperbola opens horizontally.

∴ center (h, k) = (1, -1)

foci $(h \pm c, k) = (1 \pm 5, -1)$

vertices $(h \pm a, k) = (1 \pm 4, -1)$

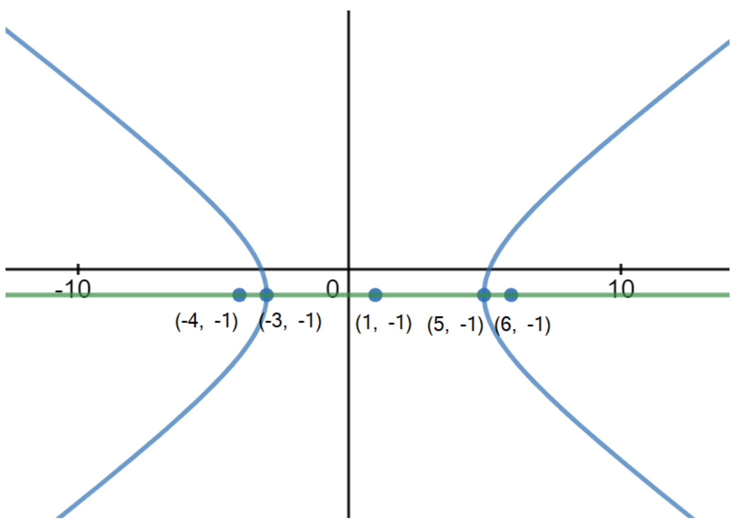
eccentricity $(e) = \frac{c}{a} = \frac{5}{4}$

length of transverse axis = $2a = 2 \times 4 = 8$

length of conjugate axis = $2b = 2 \times 3 = 6$

directrix $x = h \pm \frac{a}{e} \Rightarrow x = 1 \pm \frac{4}{(5/4)} \Rightarrow x = 1 \pm \frac{16}{5}$

The graph of this hyperbola is as follows:



Illustration

Find the equation of the hyperbola whose focus is at (6, 0), directrix $4x - 3y = 6$ and eccentricity is $e = 5/4$.

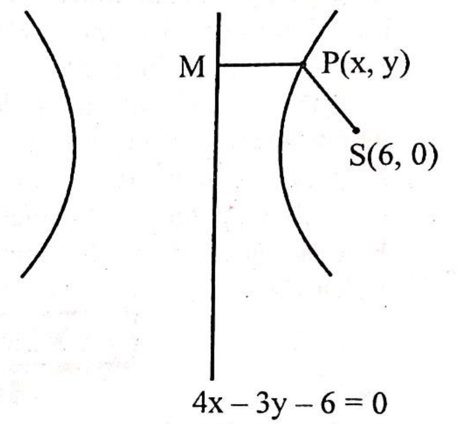
Solution

Let P(x, y) be any point on the hyperbola and let PM be the perpendicular drawn to the directrix. Let the focus F be at (6, 0).

By definition

$$e = \frac{\text{Distance between P and F}}{\text{Perpendicular distance from P to directrix}}$$

i.e., $$\frac{5}{4} = \frac{\sqrt{(x-6)^2+(y-0)^2}}{\left(\pm\frac{4x-3y-6}{\sqrt{4^2+3^2}}\right)}$$



Squaring both sides, we get

$$\frac{25}{16} \times \frac{1}{25}(4x-3y-6)^2 = (x-6)^2 + (y-0)^2$$

or, $$(4x-3y-6)^2 = 16[(x-6)^2 + y^2]$$

or, $$7y^2 + 24xy - 144x - 36y + 540 = 0$$

which is required equation of hyperbola.

Exercise for Reader

1. Find the center, foci and vertices of the hyperbola $16y^2 - 9x^2 = 144$. Sketch the graph.
2. Find the center, foci, vertices, eccentricity, length of transverse and conjugate axes, and directrix of the hyperbola $5x^2 - 4y^2 + 20x + 8y = 4$.
3. Find the center, foci, vertices, eccentricity, length of transverse and conjugate axes, and directrix of the hyperbola $2x^2 - y^2 + 6y = 3$.
4. Find the equation of the hyperbola whose focus is at (2, 1), directrix $x + 2y = 1$ and eccentricity is $e = \sqrt{2}$.