

Lecture 26

Learning Objectives

At the end of this class, students should be able to:

- find the equation of tangent line to the given ellipse
- find the equation of normal to the given ellipse
- solve related problems

Equation of Tangent

The equation of ellipse is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

$$\text{or, } b^2x^2 + a^2y^2 = a^2b^2$$

Differentiating with respect to x, we get

$$2b^2x + 2a^2y \frac{dy}{dx} = 0$$

$$\text{or, } \frac{dy}{dx} = -\frac{b^2x}{a^2y}$$

At (x_1, y_1) ,

$$\frac{dy}{dx} = -\frac{b^2x_1}{a^2y_1}$$

$$\therefore \text{ slope of the tangent } m = -\frac{b^2x_1}{a^2y_1}$$

Equation of tangent passing through (x_1, y_1) having slope m is

$$y - y_1 = m(x - x_1)$$

$$\text{or, } y - y_1 = -\frac{b^2x_1}{a^2y_1}(x - x_1)$$

$$\text{or, } a^2yy_1 - a^2y_1^2 = -b^2xx_1 + b^2x_1^2$$

$$\text{or, } b^2xx_1 + a^2yy_1 = a^2y_1^2 + b^2x_1^2$$

Dividing by a^2b^2 , we get

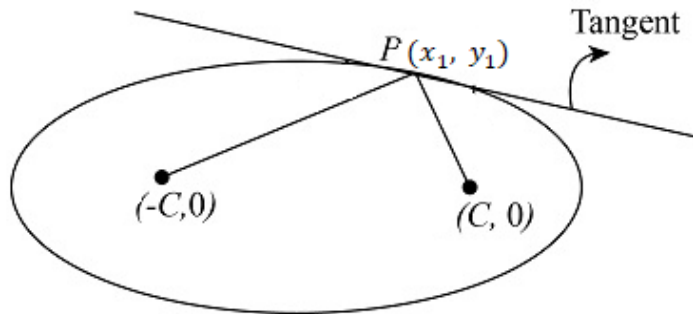
$$\frac{xx_1}{a^2} + \frac{yy_1}{b^2} = \frac{x_1^2}{a^2} + \frac{y_1^2}{b^2}$$

Since (x_1, y_1) lies on ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. Therefore

$$\frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} = 1$$

\therefore The required equation of tangent is

$$\frac{xx_1}{a^2} + \frac{yy_1}{b^2} = 1$$



Equation of Normal

The equation of tangent to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is $\frac{xx_1}{a^2} + \frac{yy_1}{b^2} = 1$.

The slope of tangent is $m = -\frac{b^2x_1}{a^2y_1}$

Thus, the slope of normal is $\frac{a^2y_1}{b^2x_1}$

Hence, the equation of normal at (x_1, y_1) is

$$y - y_1 = \frac{a^2y_1}{b^2x_1}(x - x_1)$$

or, $b^2x_1(y - y_1) = a^2y_1(x - x_1)$

or, $\frac{a^2(x-x_1)}{x_1} = \frac{b^2(y-y_1)}{y_1}$

Which is required

Condition for Tangency

Let the equation of line and ellipse are

$$y = mx + c \quad \text{(i)}$$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad \text{(ii)}$$

For the point of interaction, we solve equations (i) and (ii) simultaneously.

From (i) and (ii), we have

$$\frac{x^2}{a^2} + \frac{(mx+c)^2}{b^2} = 1$$

or, $b^2x^2 + a^2(m^2x^2 + 2cmx + c^2) = a^2b^2$

or, $(b^2 + a^2m^2)x^2 + 2a^2mcx + (a^2c^2 - a^2b^2) = 0$ (iii)

It is quadratic in x . It has two roots. If the line $y = mx + c$ touches the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, the two roots must be equal. The condition for this is

Discriminant of the equation (iii) must be zero

i.e., $(2a^2mc)^2 - 4(b^2 + a^2m^2)(a^2c^2 - a^2b^2) = 0$

or, $4a^4m^2c^2 - 4a^2(b^2c^2 + a^2c^2m^2 - b^4 - a^2b^2m^2) = 0$

or, $4a^2(a^2m^2c^2 - b^2c^2 - a^2c^2m^2 + b^4 + a^2b^2m^2) = 0$

or, $-b^2c^2 + b^4 + a^2b^2m^2 = 0$

or, $c^2 = a^2m^2 + b^2$ which is the required condition for tangency.

or, $c = \pm\sqrt{a^2m^2 + b^2}$

\therefore the line (i) is tangent to ellipse (ii) if $c = \pm\sqrt{a^2m^2 + b^2}$

Thus, $y = mx \pm \sqrt{a^2m^2 + b^2}$ is the required equation of tangent.

Illustration

Find the equation of tangent and normal at the point (4, 3) on the ellipse $3x^2 + 4y^2 = 84$.

Solution

Here, ellipse is $3x^2 + 4y^2 = 84$

or, $\frac{x^2}{27} + \frac{y^2}{21} = 1$

Now, the equation of tangent to the ellipse $\frac{x^2}{27} + \frac{y^2}{21} = 1$ at the point (4, 3) is

$$\frac{4x}{27} + \frac{3y}{21} = 1 \quad \left[\text{using } \frac{xx_1}{a^2} + \frac{yy_1}{b^2} = 1 \right]$$

or, $12x + 12y = 84$

or, $x + y = 7$

Slope of the tangent is -1, therefore, the slope of normal will be 1.

Equation of normal through the point (4, 3) is

$$y - 3 = x - 4 \quad \left[\text{using } y - y_1 = m(x - x_1) \right]$$

or, $x - y = 1$

Illustration

Find the condition that the line $lx + my + n = 0$ may touch the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. Also find the point of contact.

Solution

We know that the equation of tangent to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ at the point (x_1, y_1) is

$$\frac{xx_1}{a^2} + \frac{yy_1}{b^2} = 1 \quad \text{(i)}$$

If the line $lx + my + n = 0$ (ii)

touches the ellipse at the same point, then equations (i) and (ii) must be identical.

Comparing the coefficients of like terms in (i) and (ii), we get

$$\frac{\frac{x_1}{a^2}}{l} = \frac{\frac{y_1}{b^2}}{m} = \frac{-1}{n}$$

or, $\frac{x_1}{a^2l} = \frac{y_1}{b^2m} = \frac{-1}{n}$

$\therefore x_1 = -\frac{a^2l}{n}$ and $y_1 = -\frac{b^2m}{n}$

But the point (x_1, y_1) lies on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, so

$$\frac{a^4 l^2}{n^2 a^2} + \frac{b^4 m^2}{n^2 b^2} = 1$$

or, $\frac{a^2 l^2}{n^2} + \frac{b^2 m^2}{n^2} = 1$

or, $a^2 l^2 + b^2 m^2 = n^2$

or, $n^2 = a^2 l^2 + b^2 m^2$

Which is the required condition.

the point of contact is $\left(-\frac{a^2 l}{n}, -\frac{b^2 m}{n}\right)$.

Illustration

Find the equation of tangents to the ellipse $x^2 + 3y^2 = 3$ which are perpendicular to the line $x = 3y + 5$.

Solution

Here, ellipse is $x^2 + 3y^2 = 3$

It can be written as $\frac{x^2}{3} + \frac{y^2}{1} = 1$ which is of the form $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

Equation of any tangent to the ellipse is $y = mx \pm \sqrt{a^2 m^2 + b^2}$

The slope of the given line is $x = 3y + 5$ is $1/3$.

Since the tangent is perpendicular to the given line $x = 3y + 5$, therefore, the slope of tangent is -3 .

\therefore Required equation of tangents are $y = -3x \pm \sqrt{3 \times 3^2 + 1}$

i.e., $y = -3x \pm 2\sqrt{7}$

Exercise for Reader

1. Find the equation of tangent and normal at the point $(-5, 3)$ on the ellipse $2x^2 + 3y^2 = 84$.
2. Show that the line $y = x \pm \frac{\sqrt{5}}{\sqrt{6}}$ touches the ellipse $2x^2 + 3y^2 = 1$. Also find the point of contact.
3. Find the equation of tangents to the ellipse $4x^2 + 3y^2 = 5$ which are parallel to the line $y = 3x + 7$.