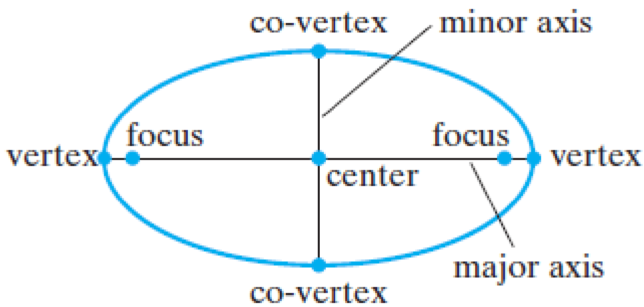
Lecture 25

Learning Objectives
At the end of this class, students should be able to:

- derive the equation of ellipse
- solve related problems

Ellipse

An ellipse is the set of points in a plane such that the sum of the distances from each point to two fixed points is constant. Each of the two fixed points is called a focus and together they are called foci. The line containing the foci intersects the ellipse at points called vertices (singular, vertex). The line segment between the vertices is called the major axis, and its midpoint is the center of the ellipse. A line perpendicular to the major axis through the center intersects the ellipse at points called the co-vertices, and the line segment between the co-vertices is called the minor axis.



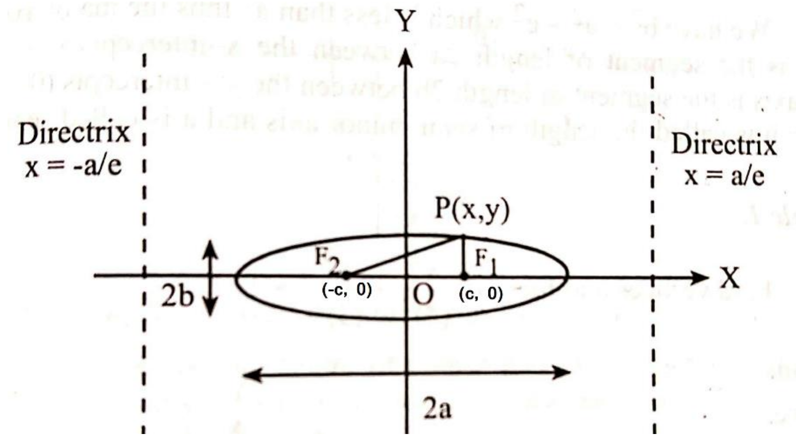
Ellipse in Architecture

Standard Equation of Ellipse

Let $F_1(c, 0)$ and $F_2(-c, 0)$ be foci of the ellipse and let $P(x, y)$ be any point of the ellipse. The sum of distance $PF_1 + PF_2$ is denoted by $2a$.



i.e., $\quad PF_1 + PF_2 = 2a$

or, $\quad \sqrt{(x-c)^2 + y^2} + \sqrt{(x+c)^2 + y^2} = 2a$

or, $\quad \sqrt{(x-c)^2 + y^2} = 2a - \sqrt{(x+c)^2 + y^2}$

Squaring both sides, we get

$$(x-c)^2 + y^2 = \left(2a - \sqrt{(x+c)^2 + y^2}\right)^2$$

or, $\quad x^2 - 2cx + c^2 + y^2 = 4a^2 + x^2 + 2cx + c^2 + y^2 - 4a\sqrt{(x+c)^2 + y^2}$

or, $\quad 4a\sqrt{(x+c)^2 + y^2} = 4(a^2 + cx)$

Again, squaring both sides, we get

$$a^2(x^2 + 2cx + c^2 + y^2) = a^4 + 2a^2cx + x^2c^2$$

or, $\quad (a^2 - c^2)x^2 + a^2y^2 = a^2(a^2 - c^2)$

or, $\frac{x^2}{a^2} + \frac{y^2}{(a^2-c^2)} = 1$ (i)

From figure, we have

$PF_1 + PF_2 > F_1F_2$

i.e., $2a > 2c$

$\Rightarrow$ $a > c$

Thus, $a^2 - c^2$ is positive and has a real positive square root and we denote $\sqrt{a^2 - c^2}$ by b.

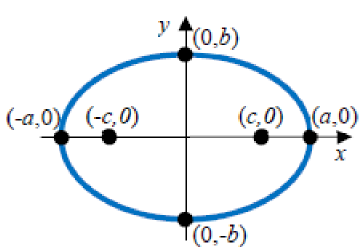
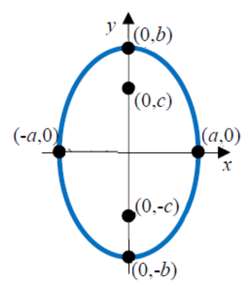
Thus, equation (i) becomes

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

Which is the equation of ellipse having center (0, 0) and foci at $(\pm c, 0)$ where $b = \sqrt{a^2 - c^2}$.

The eccentricity of the ellipse is defined by $e = \frac{c}{a} = \frac{\sqrt{a^2-b^2}}{a}$. Thus, eccentricity is the ratio of distance of focus from origin to the distance of vertex from the center. Since $a > c$ so eccentricity is less than 1 for ellipse.

Equation of an Ellipse Centered at the Origin in Standard Form

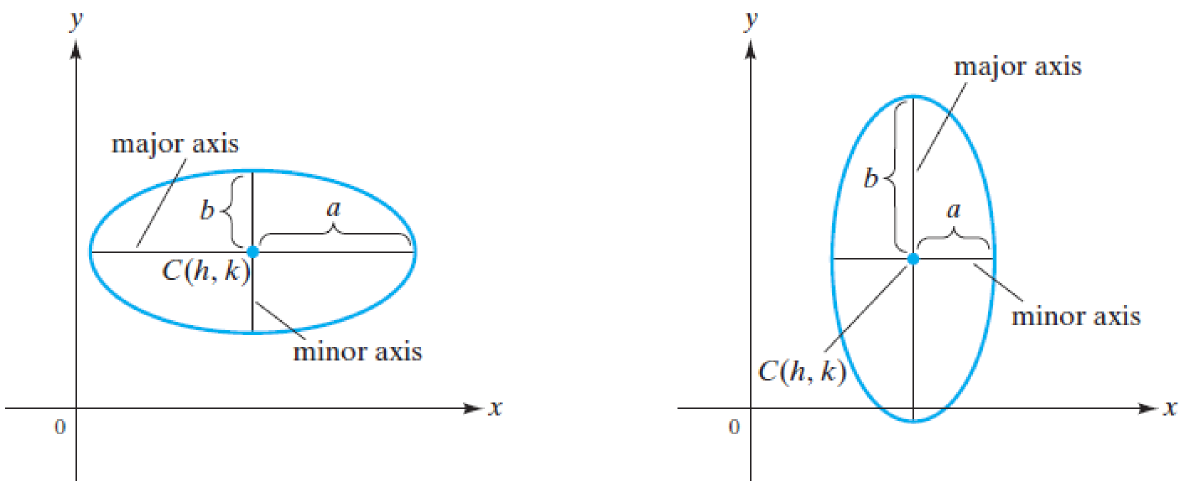
Major Axis	Horizontal	Vertical
Standard Equation	$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ for $a > b$	$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ for $a < b$
Graph		
Vertices	$(-a, 0)$ and $(a, 0)$	$(0, -b)$ and $(0, b)$
Minor axis endpoints	$(0, -b)$ and $(0, b)$	$(-a, 0)$ and $(a, 0)$
Foci	$(\pm c, 0) = (\pm ae, 0)$	$(0, \pm c) = (0, \pm be)$
Directrix	$x = \pm\frac{a}{e}$	$y = \pm\frac{b}{e}$
Length of Latus Rectum	$\frac{2b^2}{a}$	$\frac{2a^2}{b}$
Length	Major axis = $2a$ Minor axis = $2b$	Major axis = $2b$ Minor axis = $2a$
Eccentricity	c/a	c/b

Shifting Ellipse

The graph of an ellipse can have its center at (h, k) rather than at the origin (0, 0). Horizontal and vertical translations are accomplished by replacing x with $x - h$ and y with $y - k$ in the standard form of the ellipse's equation. Then the equation of ellipse is

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$$

Thus, the ellipse $\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$ is generated if the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is shifted right h units and up k units as shown in the following figures.



Equation of an Ellipse Centered at (*h, k*) in Standard Form

The standard form of an equation of an ellipse centered at the point C(*h*, *k*) depends on whether the major axis is horizontal or vertical. The table below gives the standard equation, vertices, minor axis endpoints, foci, length of latus rectum and eccentricity for each.

Major Axis	Horizontal	Vertical
Standard Equation	$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$ for $a > b$	$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$ for $a < b$
Vertices	$(h \pm a, k)$	$(h, k \pm b)$
Minor axis endpoints	$(h, k \pm b)$	$(h \pm a, k)$
Foci	$(h \pm c, k)$	$(h, k \pm c)$
Length of Latus Rectum	$\frac{2b^2}{a}$	$\frac{2a^2}{b}$
Eccentricity	c/a	c/b

Illustration

Find the equation of ellipse which has center (2, 2) focus (-1, 2) and semi-major axis $(a) = \sqrt{10}$. Also calculate eccentricity.

Solution

Here, the *y* coordinate has same value and *x* coordinate varies. So, the major axis is parallel to x-axis.

According to question,

center (h, k) = (2, 2) and

focus $(h + c, k)$ = (-1, 2)

Thus, we have $h = 2, k = 2$ and *c* = -3 (neglecting negative sign)

We know that $b^2 = a^2 - c^2$ $[\because a > b]$

$\therefore$ $b^2 = 10 - 9 = 1 \Rightarrow b = 1$

Now, the equation of an ellipse with center at (h, k) is $\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$.

i.e., $\frac{(x-2)^2}{10} + \frac{(y-2)^2}{1} = 1$

Its eccentricity $(e) = \frac{c}{a} = \frac{3}{\sqrt{10}}$

Illustration

Find the center, foci, vertices, eccentricity, and length of latus rectum of the ellipse $4x^2 + y^2 - 16x + 4y + 16 = 0$.

Solution

we have $4x^2 + y^2 - 16x + 4y + 16 = 0$

or, $4(x^2 - 4x) + (y^2 + 4y) + 16 = 0$

or, $4(x^2 - 4x + 4) + (y^2 + 4y + 4) = 4$

or, $4(x - 2)^2 + (y + 2)^2 = 4$

or, $\frac{(x-2)^2}{1^2} + \frac{(y+2)^2}{2^2} = 1$

Now, comparing with $\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$, we get

$h = 2, k = -2, a = 1, b = 2$

Here, $b > a$, so the major axis is parallel to y-axis and $c = \sqrt{b^2 - a^2} = \sqrt{4 - 1} = \sqrt{3}$

$\therefore$ center (h, k) = (2, -2)

foci $(h, k \pm c) = (2, -2 \pm \sqrt{3})$

vertices $(h, k \pm b) = (2, -2 \pm 2)$

eccentricity $(e) = \frac{c}{a} = \frac{\sqrt{3}}{1}$

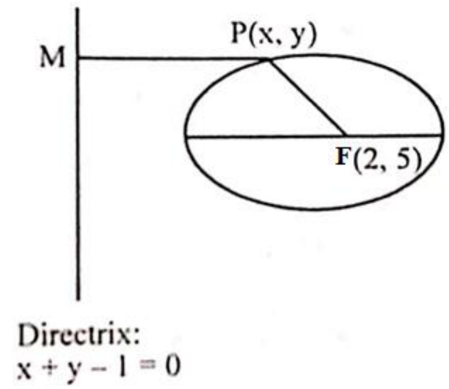
length of latus rectum = $\frac{2a^2}{b} = \frac{2\times1}{2} = 1$

Illustration

Find the equation of the ellipse whose focus is at (2, 5), directrix $x + y = 1$ and eccentricity is e = 2/3.

Solution

Let P(x, y) be any point on the ellipse and let PM be the perpendicular drawn to the directrix. Let the focus F be at (2, 5).



By definition

$$e = \frac{\text{Distance between P and F}}{\text{Perpendicular distance from P to directrix}}$$

i.e., $\frac{2}{3} = \frac{\sqrt{(x-2)^2+(y-5)^2}}{\left(\pm\frac{x+y-1}{\sqrt{1^2+1^2}}\right)}$

Squaring both sides, we get

$$\frac{4}{9}\times\frac{1}{2}(x+y-1)^2 = (x-2)^2+(y-5)^2$$

or, $7x^2 + 7y^2 - 4xy - 32x - 86y + 259 = 0$

which is required.

Exercise for Reader

1. Find the equation of ellipse which has center (0, 2) focus (0, 0) and semi-major axis $(b) = 3$. Also calculate eccentricity.
2. The end points of the major and minor axes of ellipse are (1, 1), (3, 4), (1, 7) and (-1, 4). Sketch the ellipse. Find equation of ellipse and find its focus.
3. Find the center, foci, vertices, eccentricity, and length of latus rectum of the ellipse $9x^2 + 16y^2 + 18x - 96y + 9 = 0$.
4. Find the equation of the ellipse whose focus is at (-1, 1), directrix $x - y + 3 = 0$ and eccentricity is e = 1/2.