Lecture 24

Learning Objectives

At the end of this class, students should be able to:

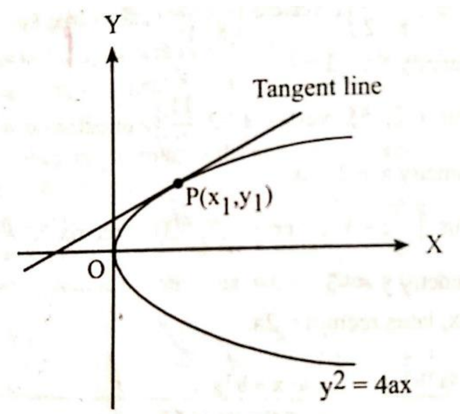
- find the equation of tangent line to the given parabola
- find the equation of normal to the given parabola
- solve related problems

Equation of Tangent

Let the equation of parabola be $y^2 = 4ax$ (i)

Let the equation of tangent at point $P(x_1, y_1)$ on the parabola given by equation (i) be

$$y = mx + c \quad \text{(ii)}$$



Then $y_1^2 = 4ax_1$ (iii)

Differentiating equation (i) with respect to x, we get

$$2y\frac{dy}{dx} = 4a$$

or, $\frac{dy}{dx} = \frac{2a}{y}$

At (x_1, y_1), $\frac{dy}{dx} = \frac{2a}{y_1} = m$ (slope of the tangent)

and $c = y_1 - mx_1$ [$\because y = mx + c$ passes through (x_1, y_1)]

Now, substituting the value of m and c in equation (ii), we get

$$y = \frac{2a}{y_1}x + y_1 - \frac{2a}{y_1}x_1$$

or, $yy_1 = 2ax + y_1^2 - 2ax_1$

or, $yy_1 = 2ax + 4ax_1 - 2ax_1$ [$\because y_1^2 = 4ax_1$]

or, $yy_1 = 2ax + 2ax_1$

$\therefore$ $yy_1 = 2a(x + x_1)$

This is the equation of tangent at point (x_1, y_1) on the parabola $y^2 = 4ax$.

Similarly, the equation of tangent to the parabola $x^2 = 4ay$ at the point (x_1, y_1) on the parabola is $xx_1 = 2a(y + y_1)$

Equation of Normal at $P(x_1, y_1)$ of $y^2 = 4ax$

the equation of tangent to the parabola $y^2 = 4ax$ at the point $P(x_1, y_1)$ on the parabola is

$$yy_1 = 2a(x + x_1)$$

Slope of this tangent is $\frac{2a}{y_1}$

Thus, the Slope of the normal is $-\frac{y_1}{2a}$

Hence, the equation of normal at $P(x_1,\ y_1)$ is

$$y - y_1 = -\frac{y_1}{2a}(x - x_1)$$

or, $\quad 2a(y - y_1) + y_1(x - x_1) = 0$

Condition for Tangency

Let the equation of line and parabola are

$$y = mx + c \qquad \text{(i)}$$

$$y^2 = 4ax \qquad \text{(ii)}$$

For the point of interaction, we solve equations (i) and (ii) simultaneously.

From (i) and (ii), we have

$$(mx + c)^2 = 4ax$$

or, $\quad m^2x^2 + 2cmx + c^2 - 4ax = 0$

or, $\quad m^2x^2 + 2(mc - 2a)x + c^2 = 0 \qquad$ (iii)

It is quadratic in x. It has two roots. If the line $y = mx + c$ touches the parabola, the two roots must be equal. The condition for this is

Discriminant of the equation (iii) must be zero

i.e., $\quad [2(mc - 2a)]^2 - 4m^2c^2 = 0$

or, $\quad 4(m^2c^2 - 4mca + 4a^2) - 4m^2c^2 = 0$

or, $\quad 4a^2 - 4mca = 0$

or, $\quad c = \frac{a}{m}$

Thus, the condition that the line $y = mx + c$touches the parabola $y^2 = 4ax$ is $c = \frac{a}{m}$.

Illustration

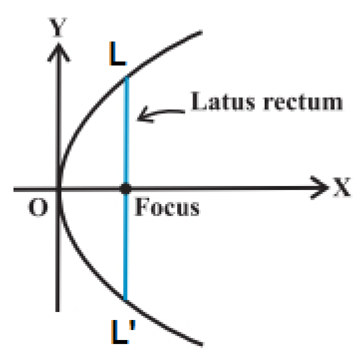
Find the equation of tangent and normal at the extremities of the latus rectum of the parabola $y^2 = 12x$.

Solution

Here, we have $y^2 = 12x$. Comparing it with $y^2 = 4ax$, we get $a = 3$.

The coordinates of the extremities of the latus rectum of the parabola $y^2 = 12x$ are $(a, 2a)$ and $(a, -2a)$.

∴ The coordinates of L are $(a, 2a) = (3, 6)$ and the coordinates of L' are $(a, -2a) = (3, -6)$.



Equation of tangent at (3, 6) is

$$y \times 6 = 2 \times 3(x + 3) \qquad [\because yy_1 = 2a(x + x_1)]$$

or, $\quad y = x + 3$

Equation of tangent at (3, -6) is

$$y \times (-6) = 2 \times 3(x + 3)$$

or, $x + y + 3 = 0$

The slope of the tangent at L(3, 6) is 1, therefore, the slope of normal at L(3, 6) is -1.

Equation of normal at (3, 6) is

$$y - 6 = (-1)(x - 3) \qquad \text{[using } y - y_1 = m(x - x_1)\text{]}$$

or, $x + y - 9 = 0$

The slope of the tangent at L'(3, -6) is -1, therefore, the slope of normal at L'(3, -6) is 1.

Equation of normal at (3, -6) is

$$y + 6 = (1)(x - 3) \qquad \text{[using } y - y_1 = m(x - x_1)\text{]}$$

or, $x - y - 9 = 0$

Illustration

Find the equation of tangent to the parabola $y^2 = 8x$ which is parallel to the straight line $2x - 3y + 1 = 0$. Also, find the point of contact.

Solution

Here, we have $y^2 = 8x$. Comparing it with $y^2 = 4ax$, we get $a = 2$.

Equation of any line parallel to the straight line $2x - 3y + 1 = 0$ is $2x - 3y + k = 0$

It can be written as $y = \frac{2}{3}x + \frac{k}{3}$ which is of the form $y = mx + c$, where $m = \frac{2}{3}$ and $c = \frac{k}{3}$.

If it is tangent to the parabola, then

$$c = \frac{a}{m}$$

i.e., $\frac{k}{3} = \frac{2}{2/3} \Rightarrow k = 9$

∴ The equation of tangent is $2x - 3y + 9 = 0$

To find the point of contact, we solve the following equations simultaneously

$$2x - 3y + 9 = 0$$

$$y^2 = 8x$$

From above equations, we get

$$\left(\frac{2x+9}{3}\right)^2 = 8x$$

or, $4x^2 - 36x + 81 = 0$

∴ $x = 9/2$

Substituting the value of x in $2x - 3y + 9 = 0$, we get $y = 6$.

$\therefore$ point of contact = (9/2, 6)

Illustration

Find the equation of common tangent to the parabolas $y^2 = 4ax$ and $x^2 = 4by$.

Solution

Equation of tangent to the parabola $y^2 = 4ax$ is $y = mx + \frac{a}{m}$ (i)

Equation of another parabola is $x^2 = 4by$ (ii)

From equation (i) and (ii), we get

$$x^2 = 4b\left(mx + \frac{a}{m}\right)$$

or, $mx^2 - 4bm^2x - 4ab = 0$ (iii)

Since the line $y = mx + \frac{a}{m}$ touches the parabola $x^2 = 4by$, therefore, the discriminant of equation (iii) must be zero.

$$(-4bm^2)^2 - 4 \times m \times (-4ab) = 0$$

or, $16m^4b^2 + 16mab = 0$

or, $m^3 = -\frac{a}{b}$

or, $m = -\frac{a^{1/3}}{b^{1/3}}$

Hence the required common tangent is

$$y = -\frac{a^{1/3}}{b^{1/3}}x + a\left(-\frac{b^{1/3}}{a^{1/3}}\right)$$

or, $y = -\frac{a^{1/3}}{b^{1/3}}x - a^{2/3}b^{1/3}$

or, $b^{1/3}y = -a^{1/3}x - a^{2/3}b^{2/3}$

or, $a^{\frac{1}{3}}x + b^{1/3}y + a^{2/3}b^{2/3} = 0$

Exercise for Reader

1. Find the condition that the line $y = mx + c$ touches the parabola $y^2 = 8x$.
2. Find the equation of tangent to the parabola $y^2 = 5x$ which is parallel to the straight line $y = 4x + 1$. Also, find the point of contact.
3. Find the condition that the line $y = mx + c$ may be tangent to the parabola $y^2 = 4ax$.