Lecture 23

Learning Objectives

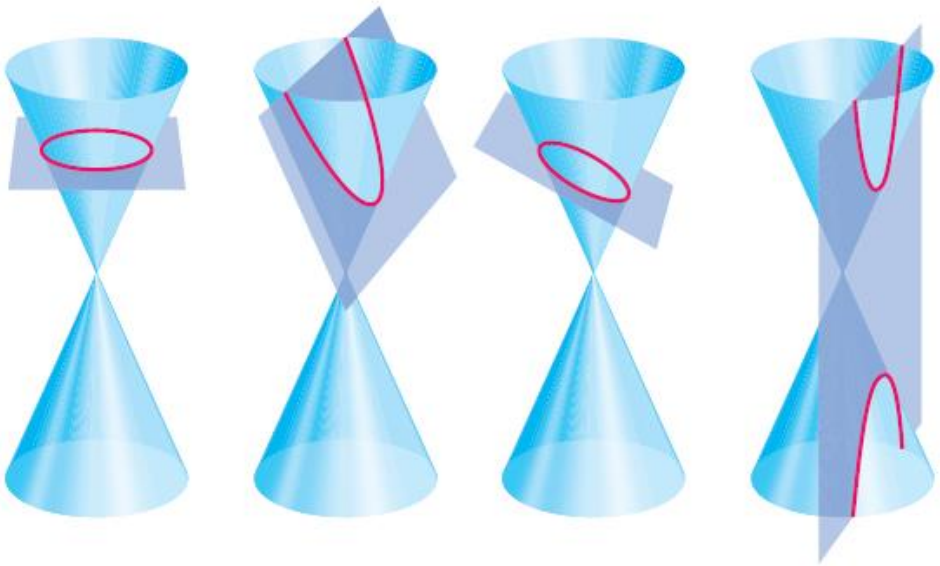
At the end of this class, students should be able to:

- understand the concept of conic sections
- derive the equation of parabola
- solve related problems

Conic sections

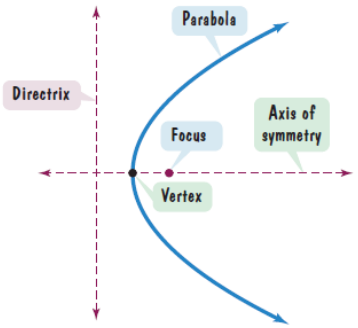
If we slice through a right circular cone with a plane, we get a variety of curves in the plane. These curves are called conic sections, which are the red lines in the following diagrams.

A plane perpendicular to the cone's axis cuts out a circle. A plane parallel to a side of the cone produces a parabola. A plane at an arbitrary angle to the axis of the cone forms an ellipse. A plane parallel to the axis cuts out a hyperbola. If we extend the cone through its vertex and form a second cone, we find the second branch of the hyperbola.



Parabola

A parabola is the set of all points in a plane equidistant from a particular line (the directrix) and a particular point (the focus) in the plane.



The line passing through the focus and perpendicular to the directrix is known as the axis of symmetry of the parabola. The point of intersection of the parabola with its axis of symmetry is called the vertex. The vertex lies in the middle between the focus and the directrix. The distance between a point on a parabola and

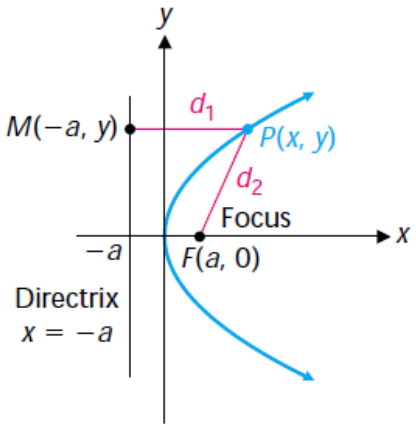
its focus is called focal distance. The line segment that passes through focus and perpendicular to axis of parabola is known as latus rectum.

Parabolas in Architecture




Equation of Parabola in Standard Form

Let the vertex of the parabola is at the origin and its axis lies along x-axis as shown in the following figure.



Let F(a, 0) be the focus of the parabola. Let P(x, y) be any point on the parabola. Join PF and draw PM perpendicular to the directrix. Since the vertex is at origin and the directrix is $x = -a$ then the coordinates of M are $(-a, y)$.

By definition of parabola,

PM = PF

i.e., $\sqrt{(x+a)^2 + (y-y)^2} = \sqrt{(x-a)^2 + (y-0)^2}$

Squaring on both sides, we get

$$(x+a)^2 = (x-a)^2 + y^2$$

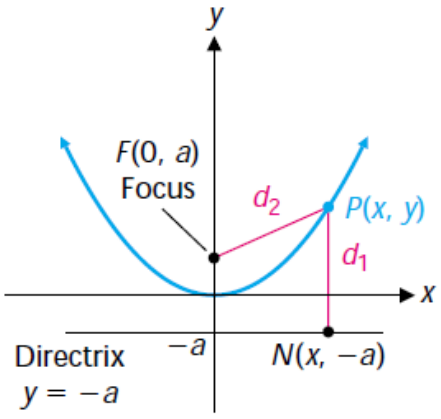
or, $x^2 + 2ax + a^2 = x^2 - 2ax + a^2 + y^2$

or, $y^2 = 4ax$

Which is the required equation of the parabola.

The eccentricity of a parabola is the ratio of PF (distance between focus and locus of a point) and PM (distance between directrix and locus of a point). Since PF = PM so eccentricity (e) = 1.

If the vertex is at the origin and focus lies on the y axis at (0, a) then the equation of the parabola is $x^2 = 4ay$. The parabola will look as follows.

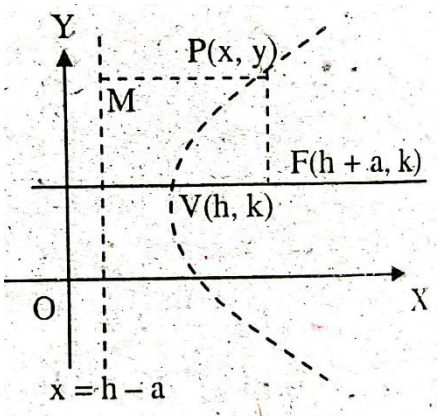


Standard Equations of a Parabola with Vertex at (0, 0)

Equation	Focus	Axis	Directrix	Opens
$y^2 = 4ax$	$(a, 0)$	x-axis	$x = -a$	to the right
$y^2 = -4ax$	$(-a, 0)$	x-axis	$x = a$	to the left
$x^2 = 4ay$	$(0, a)$	y-axis	$y = -a$	upward
$x^2 = -4ay$	$(0, -a)$	y-axis	$y = a$	downward

Shifting Parabolas

The graph of a parabola can have its vertex at (h, k) rather than at the origin (0, 0). Horizontal and vertical translations are accomplished by replacing x with $x - h$ and y with $y - k$ in the standard form of the parabola's equation.



The equation of the parabola with its axis parallel to x-axis and vertex at (h, k) is given by

$$(y - k)^2 = 4a(x - h)$$

Here, the axis of parabola is the line $y = k$. The directrix is a unit left of the vertex (h, k) and perpendicular to the axis of symmetry. Thus $x = h - a$ is the equation of directrix. Since the focus is a unit right to the vertex, therefore, the focus lies at $(h + a, k)$.

Thus, the parabola $(y-k)^2 = 4a(x-h)$ is generated if the parabola $y^2 = 4ax$ is shifted right h units and up k units as shown in the figure.

Standard Equations of a Parabola with Vertex at (h, k)

Equation	Focus	Axis of symmetry	Directrix	Opens
$(y-k)^2 = 4a(x-h)$	$(h+a, k)$	$y = k$	$x = h-a$	to the right
$(y-k)^2 = -4a(x-h)$	$(h-a, k)$	$y = k$	$x = h+a$	to the left
$(x-h)^2 = 4a(y-k)$	$(h, k+a)$	$x = h$	$y = k-a$	upward
$(x-h)^2 = -4a(y-k)$	$(h, k-a)$	$x = h$	$y = k+a$	downward

Illustration

Find the equation of the parabola if the vertex and focus lie at (2, 3) and (2, 5) respectively. Sketch the graph of the parabola.

Solution

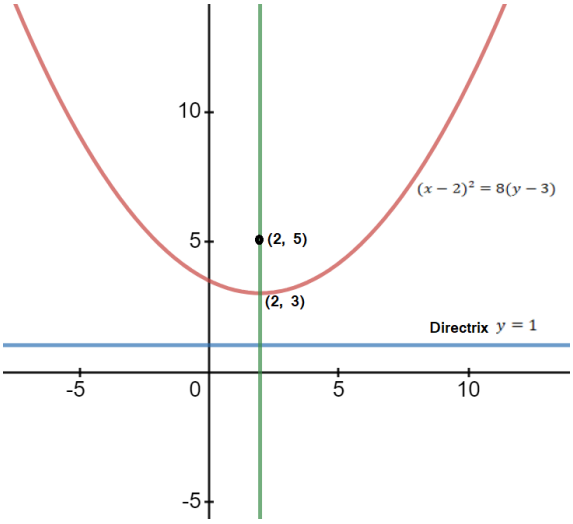
Here, the x coordinate has same value and y coordinate varies. So, the focus lies on the line parallel to y-axis.

According to question,

vertex (h, k) = (2, 3) and

focus $(h, k+a)$ = (2, 5)

Thus, $a = 2 > 0$ which indicates that the parabola opens upward.



We know that the equation of parabola is

$$(x-h)^2 = 4a(y-k)$$

i.e., $(x-2)^2 = 4 \times 2 \times (y-3)$

or, $(x-2)^2 = 8(y-3)$

Which is the required equation of parabola.

Equation of directrix is $y = k - a$, i.e., $y = 1$.

Illustration

Find the equation of the parabola if the vertex lies at (-3, 1) and directrix is $x = 1$.

Solution

Here, vertex (h, k) = (-3, 1) and directrix is $x = 1$.

We know that directrix is of the form $x = h - a$, i.e., $1 = -3 - a \Rightarrow a = -4$.

Since directrix is the line $x = 1$ so the focus and vertex both lie on the line parallel to x-axis.

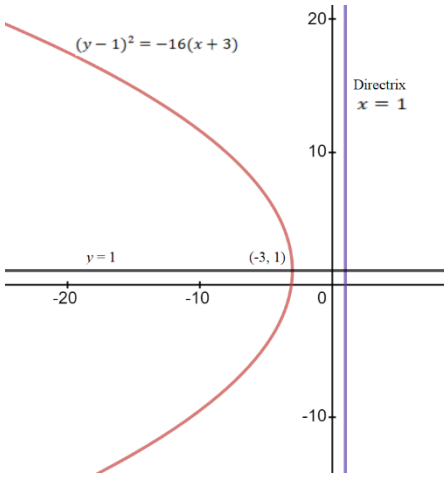
Now focus $(h+a, k)$ = $(-3-4, 1) = (-7, 1)$

The equation of parabola is

$$(y-k)^2 = 4a(x-h)$$

or, $(y-1)^2 = 4 \times (-4) \times (x+3)$

or, $(y-1)^2 = -16(x+3)$



Illustration

Find the vertex, axis of symmetry, focus, and directrix of the parabola $y + 4x^2 - 3x = 0$.

Solution

Here, $y + 4x^2 - 3x = 0$

or, $x^2 - \frac{3}{4}x = -\frac{y}{4}$

or, $x^2 - 2 \times \frac{3}{8} \times x + \left(\frac{3}{8}\right)^2 = -\frac{y}{4} + \left(\frac{3}{8}\right)^2$

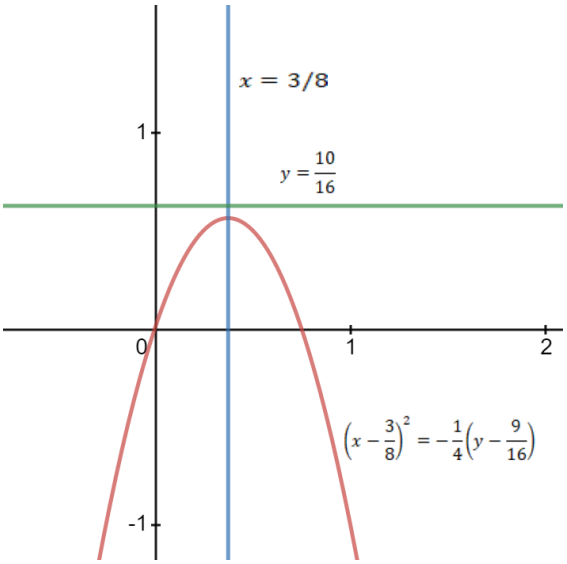
or, $\left(x - \frac{3}{8}\right)^2 = -\frac{1}{4}\left(y - \frac{9}{16}\right)$

Comparing with $(x-h)^2 = -4a(y-k)$

vertex $(h, k) = \left(\frac{3}{8}, \frac{9}{16}\right)$;

$4a = \frac{1}{4} \Rightarrow a = \frac{1}{16}$;

Focus $(h, k-a) = \left(\frac{3}{8}, \frac{9}{16} - \frac{1}{16}\right) = \left(\frac{3}{8}, \frac{8}{16}\right)$;



Axis of symmetry is given by $x = h$, i.e., $x = 3/8$;

Equation of directrix is $y = k + a$, i.e., $y = \frac{9}{16} + \frac{1}{16} \Rightarrow y = \frac{10}{16}$.

Exercise for Reader

1. Find the equation of the parabola if the vertex and focus lie at (-3, 1) and (0, 1) respectively. Sketch the graph of the parabola.
2. Find the equation of the parabola if the vertex lies at (0, 1) and directrix is $x = -1$.
3. Find the vertex, focus, and directrix of the parabola $y^2 + 36x = 0$.
4. Find the vertex, axis of symmetry, focus, and directrix of the parabola $y^2 - 8y - x + 19 = 0$.