Lecture 22

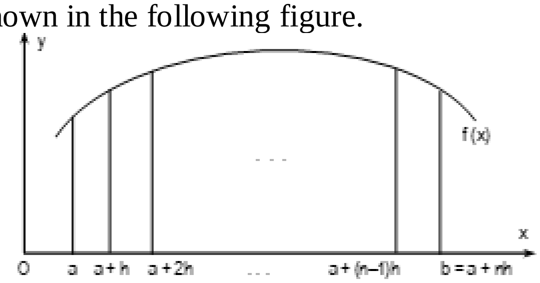
Learning Objectives

At the end of this class, students should be able to:

- understand the concept of Trapezoidal Rule
- understand the concept of Simpson's Rule
- evaluate the value of integral by using Trapezoidal Rule and Simpson's Rule

Trapezoidal Rule

This is the method of finding the integration of a function by dividing the area under a curve into trapezoids. We assume that $f(x) \geq 0$ on the interval $[a, b]$. Let the interval $[a, b]$ be divided into n equal subintervals each of length $h = \frac{b-a}{n}$. With each interval we can associate a trapezoid. Thus, there are n trapezoids as shown in the following figure.



The area of the first trapezoid $= \frac{1}{2}h[f(a) + f(a+h)]$

$$\left[\because \text{Area of trapezoid } = \frac{1}{2}(\text{Sum of parallel sides}) \times \text{Distance between them}\right]$$

The area of the second trapezoid $= \frac{1}{2}h[f(a+h) + f(a+2h)]$

Similarly, area of the last trapezoid $= \frac{1}{2}h[f\{a+(n-1)h\} + f(a+nh)]$

$$= \frac{1}{2}h[f\{a+(n-1)h\} + f(b)]$$

Therefore, the sum of the areas of these n trapezoids

$$= \frac{1}{2}h[f(a) + 2f(a+h) + 2f(a+2h) + \ldots 2f\{a+(n-1)h\} + f(b)]$$

which is the approximate area of the region under the curve $y = f(x)$, $a \leq x \leq b$.

Thus

$$\int_a^b f(x)dx \approx \frac{1}{2}h[f(a) + 2f(a+h) + 2f(a+2h) + \ldots + f(b)], \text{ where } h = \frac{b-a}{n}$$

Illustration

Use the trapezoidal rule with $n = 4$ to estimate $\int_0^2 x^2 dx$. Compare the estimate with the exact value of the integral.

Solution

We divide the interval of integration into 4 equal parts. Here, $a = 0$, $b = 2$ and $n = 4$.

$$h = \frac{b-a}{n} = \frac{2-0}{4} = \frac{2}{4} = 0.5$$

The integrand is $f(x) = x^2$. Hence

$$f(a) = f(0) = 0^2 = 0$$

$$f(a+h) = f(0+0.5) = f(0.5) = (0.5)^2 = 0.25$$

$$f(a+2h) = f(0+2\times 0.5) = f(1) = (1)^2 = 1$$

$$f(a+3h) = f(0+3\times 0.5) = f(1.5) = (1.5)^2 = 2.25$$

$$f(b) = f(2) = (2)^2 = 4$$

$$\therefore \int_0^2 x^2 dx \approx \frac{1}{2}h[f(a) + 2f(a+h) + 2f(a+2h) + 2f(a+3h) + f(b)]$$

$$= \frac{0.5}{2}[f(0) + 2f(0.5) + 2f(1) + 2f(1.5) + f(2)]$$

$$= 0.25[0 + 2\times 0.25 + 2\times 1 + 2\times 2.25 + 4]$$

$$= 2.75$$

Also, the exact value of the integral is

$$\int_0^2 x^2 dx = \frac{x^3}{3}\Big|_0^2 = \frac{8}{3} = 2.666$$

Clearly, the approximation is a slight overestimate.

Thus, the Trapezoidal expectation is $\left(\frac{2.75-\frac{8}{3}}{\frac{8}{3}}\right) \times 100\% = 3.125\%$ greater than the exact value.

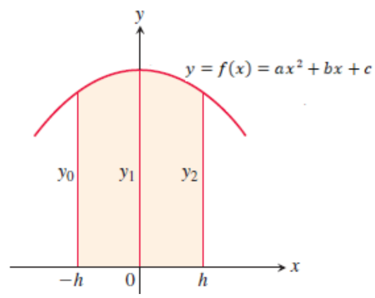
Simpson' Rule

This rule involves approximating the graph of the function $f(x)$ by parabolic segments.

The equation of the parabolic segment is of the form $y = f(x) = ax^2 + bx + c$.

The area of the shaded region under the curve from $x = -h$ to $x = h$ is given by

$$A = \int_{-h}^{h} f(x)dx = \int_{-h}^{h}(ax^2 + bx + c)dx$$



or, $A = \left[\frac{ax^3}{3} + \frac{bx^2}{2} + cx\right]_{-h}^{h}$

or, $A = \frac{2ah^2}{3} + 2ch$

or, $A = \frac{h}{3}(2ah^2 + 6c)$ (i)

Also, $y_0 = f(-h) = ah^2 - bh + c$ $[\because f(x) = ax^2 + bx + c]$

$y_1 = f(0) = c$

$y_2 = f(h) = ah^2 + bh + c$

Now, $f(-h) + 4f(0) + f(h) = 2ah^2 + 6c$ (ii)

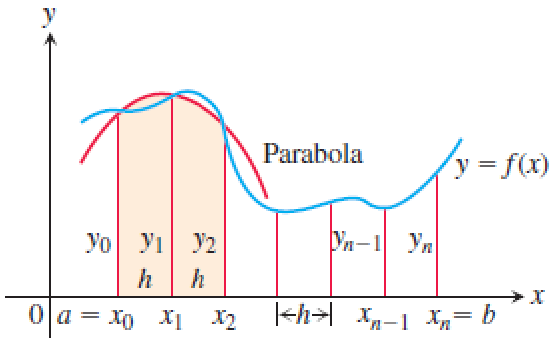
From equation (i) and (ii), we get

$$A = \frac{h}{3}[f(-h) + 4f(0) + f(h)]$$

Thus, the area of the region under the curve from $x = -h$ to $x = h$ is

$$\int_{-h}^{h}(ax^2 + bx + c)dx = \frac{h}{3}[f(-h) + 4f(0) + f(h)] \qquad \text{(iii)}$$

or, $$\int_{-h}^{h}(ax^2 + bx + c)dx = \frac{h}{3}[y_0 + 4y_1 + y_2]$$



Applying this formula of A to successive pieces of the curve $y = f(x)$ between $x = a$ to $x = b$, we get the Simpson's rule as:

$$\int_a^b f(x)dx \approx \frac{h}{3}[y_0 + 4y_1 + 2y_2 + 4y_3 + 2y_4 + \ldots + 2y_{n-2} + 4y_{n-1} + y_n]$$

or, $$\int_a^b f(x)dx \approx \frac{h}{3}[f(a) + 4f(a+h) + 2f(a+2h) + 4f(a+3h) + 2f(a+4h) + \ldots + f(b)]$$

where $h = \frac{b-a}{n}$ and n is even.

The pattern of coefficients inside the brackets is

1, 4, 2, 4, 2, 4, 2, . . . , 4, 2, 4, 1

and this requires that n be even.

Illustration

Use Simpson's rule to estimate the value of the integral $\int_0^2 x^2 dx$ by using $n = 4$.

Solution

Here, $a = 0$, $b = 2$ and $n = 4$.

$$h = \frac{b-a}{n} = \frac{2-0}{4} = \frac{2}{4} = 0.5$$

The integrand is $f(x) = x^2$. Hence

$$f(a) = f(0) = 0^2 = 0$$

$$f(a+h) = f(0+0.5) = f(0.5) = (0.5)^2 = 0.25$$

$$f(a+2h) = f(0 + 2 \times 0.5) = f(1) = (1)^2 = 1$$

$$f(a+3h) = f(0 + 3 \times 0.5) = f(1.5) = (1.5)^2 = 2.25$$

$f(b) = f(2) = (2)^2 = 4$

By Simpson’s rule

$$\int_a^b f(x)dx \approx \frac{h}{3}[f(a) + 4f(a+h) + 2f(a+2h) + 4f(a+3h) + f(b)]$$

$$\int_0^2 x^2 dx \approx \frac{0.5}{3}[f(0) + 4f(0.5) + 2f(1) + 4f(1.5) + f(2)]$$

$$= \frac{0.5}{3}[0 + 4 \times 0.25 + 2 \times 1 + 4 \times 2.25 + 4]$$

$$= \frac{0.5 \times 16}{3}$$

$$= \frac{8}{3}$$

Illustration

Approximate the integral $\int_0^\pi \sin x\, dx$ with $n = 4$ using (a) Trapezoidal rule and (b) Simpson's rule; and then compare the results with the exact value of the integral.

Solution

Here, $a = 0$, $b = \pi$ and $n = 4$.

$$h = \frac{b-a}{n} = \frac{\pi - 0}{4} = \frac{\pi}{4}$$

The integrand is $f(x) = \sin x$. Hence

$$f(a) = f(0) = \sin 0 = 0$$

$$f(a+h) = f\left(0 + \frac{\pi}{4}\right) = f\left(\frac{\pi}{4}\right) = \sin\frac{\pi}{4} = \frac{1}{\sqrt{2}}$$

$$f(a+2h) = f\left(0 + 2 \times \frac{\pi}{4}\right) = f\left(\frac{\pi}{2}\right) = \sin\frac{\pi}{2} = 1$$

$$f(a+3h) = f\left(0 + 3 \times \frac{\pi}{4}\right) = f\left(\frac{3\pi}{2}\right) = \sin\frac{3\pi}{2} = \frac{1}{\sqrt{2}}$$

$$f(b) = f(\pi) = \sin\pi = 0$$

Now, Trapezoidal rule

$$\int_0^\pi \sin x\, dx \approx \frac{1}{2}h[f(a) + 2f(a+h) + 2f(a+2h) + 2f(a+3h) + f(b)]$$

$$= \frac{(\pi/4)}{2}\left[f(0) + 2f\left(\frac{\pi}{4}\right) + 2f\left(\frac{\pi}{2}\right) + 2f\left(\frac{3\pi}{2}\right) + f(\pi)\right]$$

$$= \frac{\pi}{8}\left[0 + 2 \times \frac{1}{\sqrt{2}} + 2 \times 1 + 2 \times \frac{1}{\sqrt{2}} + 0\right]$$

$$= \frac{22}{7 \times 8}\left[2\sqrt{2} + 2\right]$$

$$= 1.8968$$

By Simpson’s rule

$$\int_a^b f(x)dx \approx \frac{h}{3}[f(a) + 4f(a+h) + 2f(a+2h) + 4f(a+3h) + f(b)]$$

$$\int_0^{\pi} \sin x \, dx = \frac{\pi}{12}\left[f(0) + 4f\left(\frac{\pi}{4}\right) + 2f\left(\frac{\pi}{2}\right) + 4f\left(\frac{3\pi}{2}\right) + f(\pi)\right]$$

$$= \frac{\pi}{12}\left[0 + 4 \times \frac{1}{\sqrt{2}} + 2 \times 1 + 4 \times \frac{1}{\sqrt{2}} + 0\right]$$

$$= 2.0053$$

Also, the exact value of the integral is

$$\int_0^{\pi} \sin x \, dx \quad = [-\cos x]|_0^{\pi} \quad = 2$$

The approximation by Trapezoidal rule is underestimated by 0.1032 and the approximation by Simpson's rule is overestimated by 0.0053. Thus, Simpson's approximation is more accurate than Trapezoidal approximation.

Exercise for Reader

1. Approximate the integral $\int_0^2 x \, dx$ with n = 4 using (a) Trapezoidal rule and (b) Simpson's rule; and then compare the results with the exact value of the integral.
2. Approximate the integral $\int_1^4 \sqrt{x} \, dx$ with n = 4 using (a) Trapezoidal rule and (b) Simpson's rule; and then compare the results with the exact value of the integral.
3. Approximate the integral $\int_0^3 x^3 \, dx$ with n = 6 using (a) Trapezoidal rule and (b) Simpson's rule.