Lecture 21

Learning Objectives

At the end of this class, students should be able to:

- compute the volume(s) of the solid(s) of revolution
- solve related problems

Volume of a "Solid of Revolution"

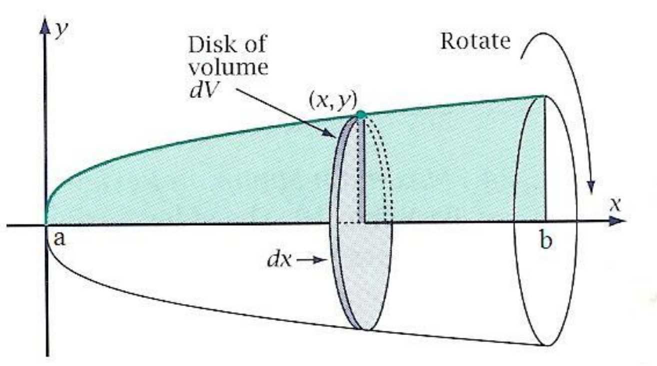
A solid of revolution is a solid figure obtained by rotating a plane area around some straight line (the axis of revolution) that lies on the same plane. For example, if a semicircle is rotated about its diameter, the solid of revolution so obtained is a sphere. If a rectangle is rotated about one of its sides, we get a right circular cylinder in a complete rotation.

Rotation about the x-Axis:

Let the area under a curve $y = f(x)$, namely, $\int_a^b f(x)\,dx$, be revolved about the x-axis, thus generating a volume. The area of a cross section of this solid by a plane perpendicular to the x-axis is πy^2 where y is the radius of the circle. The volume of a thin slice, "dx" would be $\pi y^2 dx$.

The total volume of solid of revolution between the two parallel planes $x = a$ and $x = b$ would therefore be the sum of all such slices

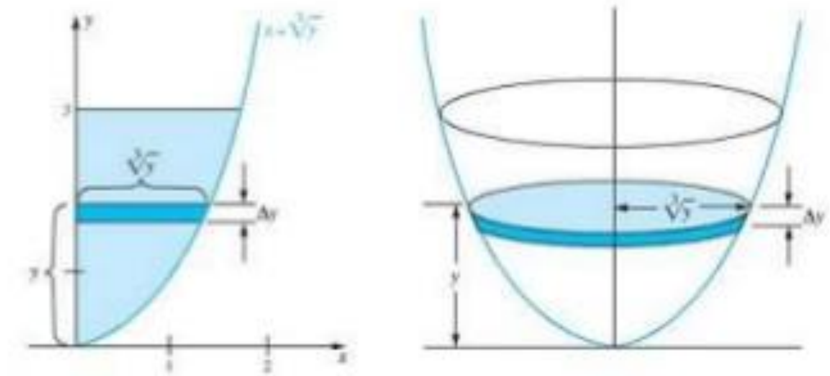
$$V = \int_a^b \pi y^2\,dx = \pi \int_a^b [f(x)]^2\,dx$$



Rotation about the y-Axis:

The volume of a solid generated by revolving the region which is between the y-axis and the curve $x = f(y), c \le y \le d$ about y-axis is

$$V = \pi \int_c^d [f(y)]^2\,dy$$



Illustration

Find the volume of the solid of revolution bounded by the curve $y = x^2$ and the lines $y = 0$ and $x = 2$ about x-axis.

Solution

The shaded region on the following figure represents the region bounded by the curve $y = x^2$, x-axis and the lines $y = 0$ and $x = 2$. Here, the axis of revolution is x-axis. Therefore, the radius function is $R(x) = x^2$ and x varies from 0 to 2.

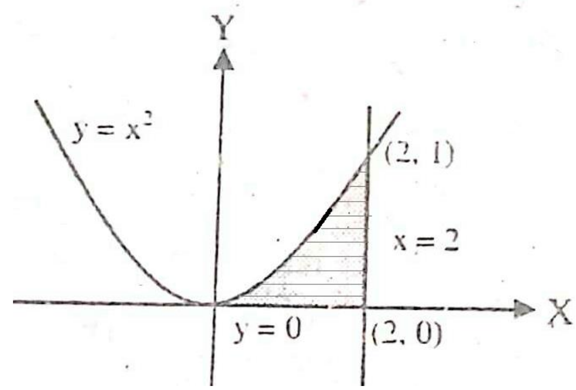
Required volume $V = \pi \int_a^b [R(x)]^2 \, dx$

$$= \pi \int_0^2 [x^2]^2 \, dx$$

$$= \pi \int_0^2 x^4 \, dx$$

$$= \pi \left[\frac{x^5}{5}\right]_0^2$$

$$= \frac{32\pi}{5} \text{ cubic units}$$



Illustration

Find the volume of the solid of revolution bounded by the curve $x = y^{3/2}$ and the lines $x = 0$ and $y = 2$ about y-axis.

Solution

The shaded region on the following figure represents the region bounded by the curve $x = y^{3/2}$ and the lines $x = 0$ and $y = 2$. Here, the axis of revolution is y-axis. Therefore, the radius function is $R(y) = y^{3/2}$ and y varies from 0 to 2.

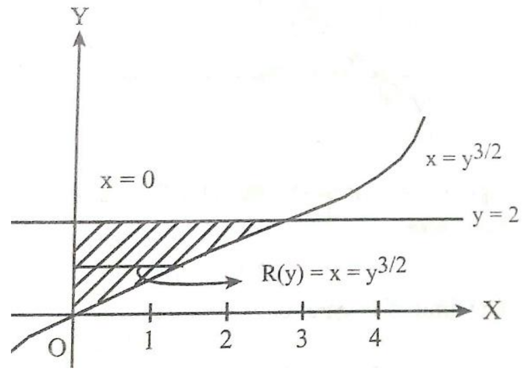
Required volume $V = \pi \int_a^b [R(y)]^2\, dy$

$= \pi \int_0^2 [y^{3/2}]^2\, dy$

$= \pi \int_0^2 y^3\, dy$

$= \pi \left[\frac{y^4}{4}\right]_0^2$

$= 4\pi$ cubic units



Illustration

Find the volume of the solid generated by revolving the region bounded by the curve $y = \sqrt{x}$ and the lines $y = 1$ and $x = 4$ about the line $y = 1$.

Solution

The shaded region on the following figure represents the region bounded by the curve $y = \sqrt{x}$ and the lines $y = 1$ and $x = 4$. Here, the axis of revolution is the line $y = 1$. Therefore, the radius function is $R(x) = \sqrt{x} - 1$and x varies from 1 to 4.

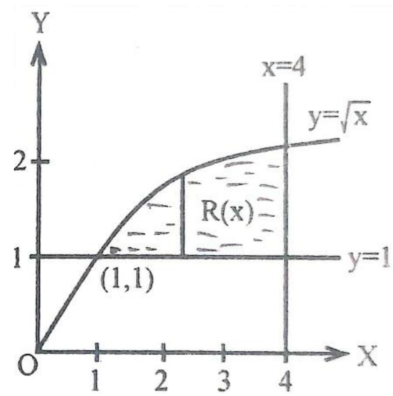
Required volume $V = \pi \int_a^b [R(x)]^2\, dx$

$= \pi \int_1^4 [\sqrt{x} - 1]^2\, dx$

$= \pi \int_1^4 \left[x - 2\sqrt{x} + 1\right] dx$

$= \pi \left[\frac{x^2}{2} - \frac{4}{3}x^{3/2} + x\right]_1^4$

$= \frac{7\pi}{6}$ cubic units



The Washer Method

When the region revolved for generating the solid doesn't border on or cross the axis of revolution, the solid has a hole in it. The cross sections perpendicular to the axis of revolution are washers instead of disks.

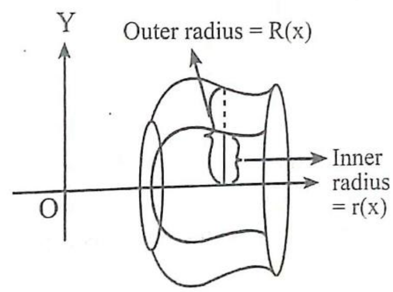
Let the dimensions of the typical washer are:

Outer radius = $R(x)$

Inner radius = $r(x)$

Thus, the washer's area is $A(x) = \pi\{R(x)\}^2 - \pi\{r(x)\}^2$

$= \pi[\{R(x)\}^2 - \{r(x)\}^2]$



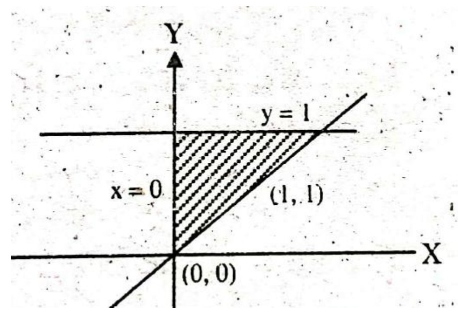
Hence, the required volume

$$V = \pi \int_a^b [\{R(x)\}^2 - \{r(x)\}^2] dx$$

Illustration

Find the volume of the solid generated by revolving the region bounded by the lines $y = x$ and the lines $y = 1$ and $x = 0$ about x-axis.

Solution



The shaded region on the following figure represents the region bounded by the lines $y = x$ and the lines $y = 1$ and $x = 0$. Here, the axis of revolution is x-axis. Here, the outer radius is $R(x) = 1$and the inner radius is $r(x) = x$ and x varies from 0 to 1.

Required volume $V = \pi \int_a^b [\{R(x)\}^2 - \{r(x)\}^2] dx$

$$= \pi \int_0^1 [1 - x^2]\, dx$$

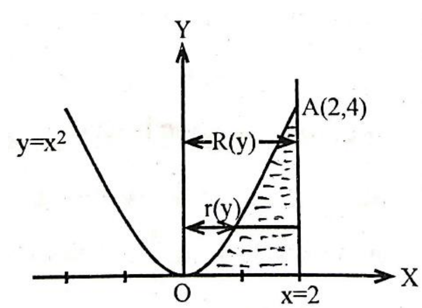
$$= \pi \left[x - \frac{x^3}{3}\right]_0^1$$

$$= \frac{2\pi}{3} \text{ cubic units}$$

Illustration

Find the volume of the solid generated by revolving the region in the first quadrant bounded above by the parabola $y = x^2$, below by x-axis and on the right by the line $x = 2$, about y-axis.

Solution



The shaded region on the following figure represents the region bounded by the parabola $y = x^2$, x-axis and the line $x = 2$. Here, the axis of revolution is y-axis. Here, the outer radius is $R(y) = 2$ and the inner radius is $r(y) = \sqrt{y}$ and y varies from 0 to 4

Required volume $V = \pi \int_c^d \left[\{R(y)\}^2 - \left\{\sqrt{y}\right\}^2\right] dy$

$$= \pi \int_0^4 [4 - y] dy$$

$$= \pi \left[4y - \frac{y^2}{2}\right]_0^4$$

$$= 8\pi \text{ cubic units}$$

Exercise for Reader

1. Find the volume of the solid of revolution bounded by the curve $y = 3x^2$ and the lines $y = 0$ and $x = 5$ about x-axis.

2. Find the volume of the solid generated by revolving the region in the first quadrant bounded by the parabola $y = x^2$, y-axis and the line $y = 2$, about y-axis.
3. Find the volume of the solid generated by revolving the region bounded by the curves $y = x^2 + 1$ and $y = x + 3$ about x-axis.
4. Show that the volume of the solid generated by revolving of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ about x-axis is $\frac{4}{3}\pi r^3$.