Lecture 20

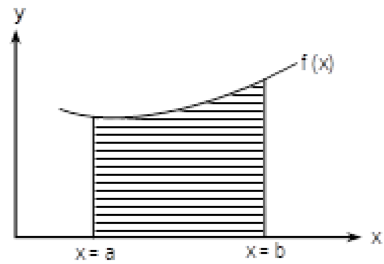
Learning Objectives

At the end of this class, students should be able to:

- find the area of plane region
- solve related problems

Area under a Curve

Let $f(x)$ is a continuous function on $[a, b]$ as given in the following figure and $f(x) \geq 0$ on $[a, b]$.



Let A be the area of a shaded region between $y = f(x)$ and the x-axis from $x = a$ to $x = b$, then

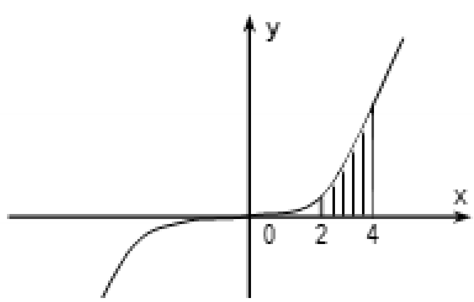
$$A = \int_a^b f(x)\, dx$$

Illustration

Find the area bounded by the curve $y = 3x^3$, x-axis and the ordinates at $x = 2$ and $x = 4$.

Solution

The given curve is $y = 3x^3$. Let A be the area bounded by the curve $y = 3x^3$, x-axis and ordinates at $x = 2$, and $x = 4$ as shown in the following figure.



Then

$$A = \int_2^4 y dx$$

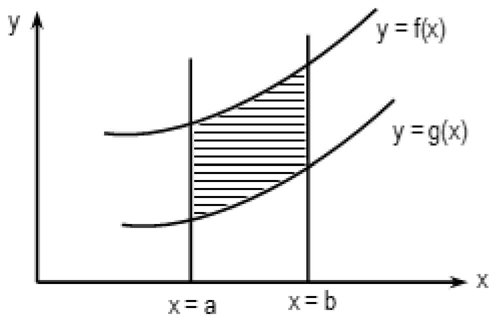
$= \int_2^4 3x^3 dx$

$= \frac{3}{4} x^4 \Big|_2^4$

$= 180$

Area between two Curves

The shaded region in the following figure is the area bounded by two curves $y = f(x)$ and $y = g(x)$, and the two ordinates at $x = a$, and $x = b$.

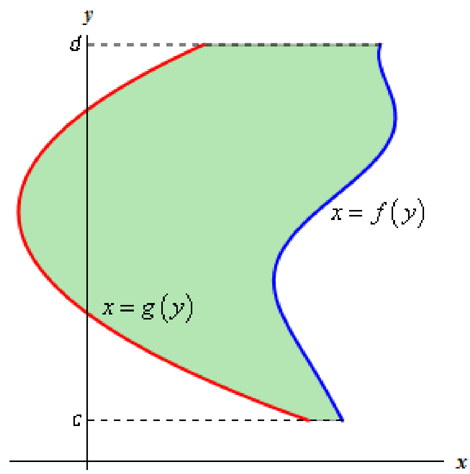


Let A be the area of the shaded region. Then

$$A = \int_a^b f(x)dx - \int_a^b g(x)dx$$

or, $$A = \int_a^b [f(x) - g(x)]dx$$

The shaded region in the following figure is the area bounded by two curves $x = f(y)$ and $x = g(y)$ in between $y = c$ and $y = d$.



Let A be the area of the shaded region. Then

$$A = \int_c^d [f(y) - g(y)]dy$$

Note:

1. If an area comes out to be negative, we reject the negative sign and consider the numerical value only. The area comes out to be negative when $f(x) < 0$, i.e., when the graph of $f(x)$ lies below the x-axis (or, $f(y) < 0$, i.e. when the graph of $f(y)$ lies left to the y-axis).
2. If we are to find the sum of areas, we shall find their numerical sum and not algebraic sum after calculating each separately.
3. If a curve is symmetrical, we find the area of one symmetrical portion and multiply it by *n* if there is *n* such symmetrical portions.

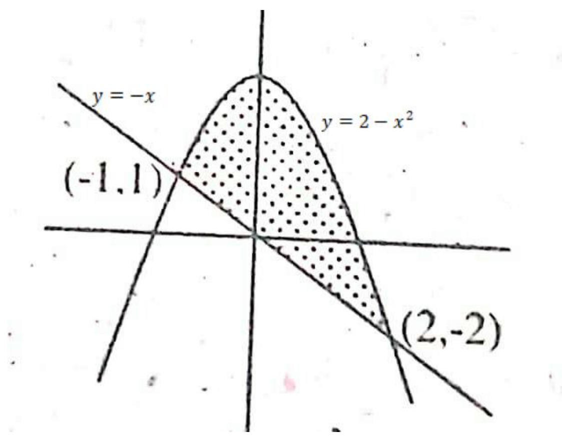
Illustration

Compute the area of the region bounded by the curve $y = 2 - x^2$ and the straight line $y = -x$.

Solution

To find the points of intersection, we solve the system of equations $y = 2 - x^2$ and $y = -x$, we get $x^2 - x - 2 = 0$ or $(x + 1)(x - 2) = 0$. Thus, the points of intersection are at $x = -1$ and $x = 2$.

The graph is shown in the following figure.



The upper portion of the shaded region is bounded by the curve $y = 2 - x^2$ and the lower portion is bounded by the straight line $y = -x$.

Here, *x* varies from -1 to 2. The desired area of the shaded region is given by

$$A = \int_a^b [f(x) - g(x)]dx$$

$$= \int_{-1}^{2} [(2 - x^2) - (-x)]dx$$

$$= \int_{-1}^{2} [2 - x^2 + x]dx$$

$$= \left[2x - \frac{x^3}{3} + x^2\right]_{-1}^{2}$$

= 9/2 square unit

Illustration

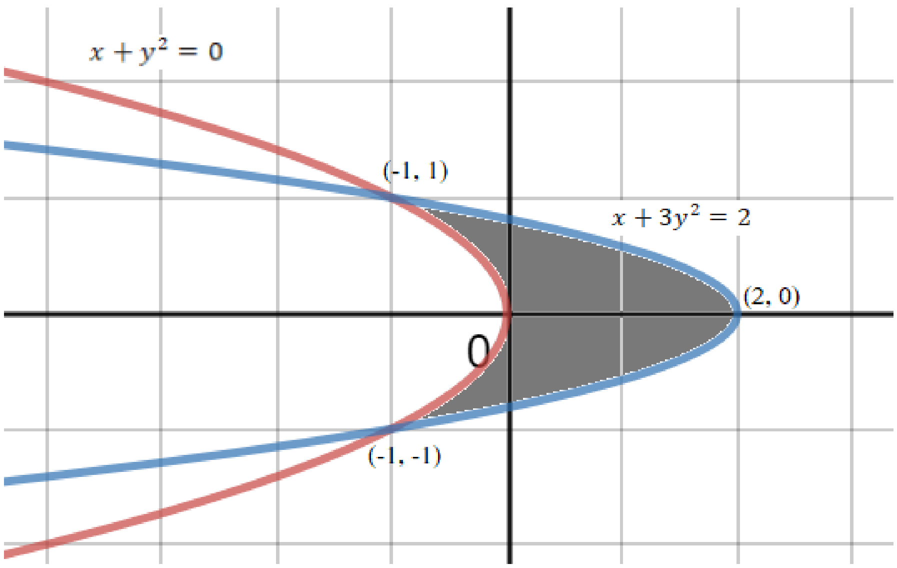
Compute the area of the region bounded by the curve $x + y^2 = 0$ and the curve $x + 3y^2 = 2$.

Solution

Here both the curves $x + y^2 = 0$ and $x + 3y^2 = 2$ are the equation of parabolas that are symmetrical with x-axis.

To find the points of intersection, we solve the system of equations $x + y^2 = 0$ and $x + 3y^2 = 2$, we get $2y^2 = 2$ or $y = \pm 1$ and $x = -1$. Thus, the points of intersection are at $(-1, -1)$ and $(-1, 1)$.

The graph is shown in the following figure.



The right portion of the shaded region is bounded by the curve $x = 2 - 3y^2$ and the left portion is bounded by the curve $x = -y^2$.

Here, y varies from -1 to 1. The desired area of the shaded region is given by

$$
\begin{aligned}
A &= \int_c^d [f(y) - g(y)]dy \\
&= \int_{-1}^{1} [(2 - 3y^2) - (-y^2)]dy \\
&= \int_{-1}^{1} [2 - 2y^2]dy \\
&= \left[2y - \frac{2y^3}{3}\right]_{-1}^{1} \\
&= 8/3 \text{ square unit}
\end{aligned}
$$

Illustration

Compute the area of the region bounded by the curve $x^2 = 4y$ and the line $y = |x|$.

Solution

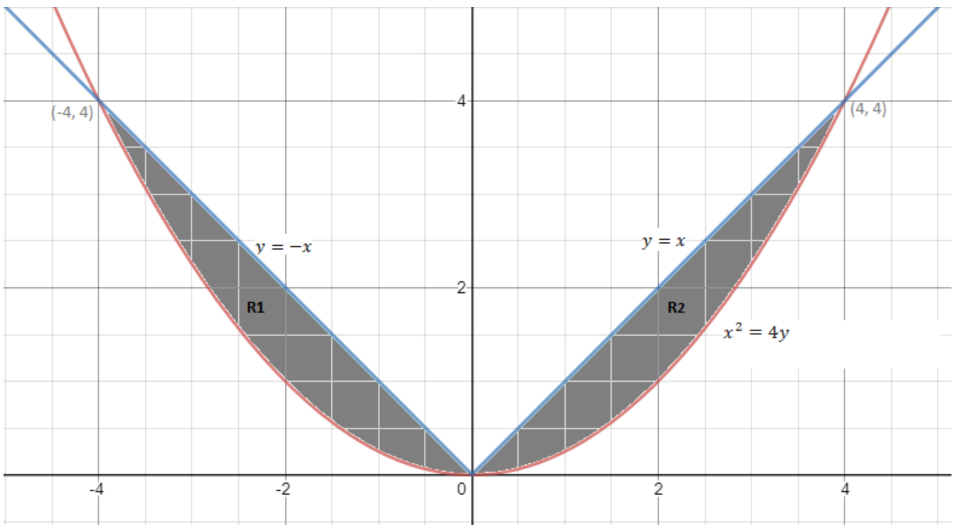
Here the curve $x^2 = 4y$ is the equation of parabola which is symmetrical with y-axis.

We know that

$$y = |x| = \begin{cases} -x & \text{if } x < 0 \\ x & \text{if } x \geq 0 \end{cases}$$

Solving the system of equations $x^2 = 4y$ and $y = -x$, we get points of intersections (0, 0) and $(-4, 4)$. Similarly, solving the system of equations $x^2 = 4y$ and $y = x$, we get points of intersections (0, 0) and $(4, 4)$.

The graph is shown in the following figure. It consists of two regions R_1 and R_2.



The upper boundary of the region R_1 is $y = -x$ and the lower boundary is $y = x^2/4$.

Here, *x* varies from -4 to 0.

The desired area of the shaded region R_1 is given by

$$A = \int_{-4}^{0} [(-x) - (x^2/4)]dx$$

$$= \int_{-4}^{0} [-x - x^2/4]dx$$

$$= \left[\frac{x^2}{2} - \frac{x^3}{12}\right]_{-4}^{0}$$

= - 8/3

= 8/3 square unit [neglecting negative sign]

Similarly, the area of region R2 is 8/3 square unit

Thus, the total area of shaded region is $\frac{8}{3} + \frac{8}{3} = \frac{16}{3}$ square unit

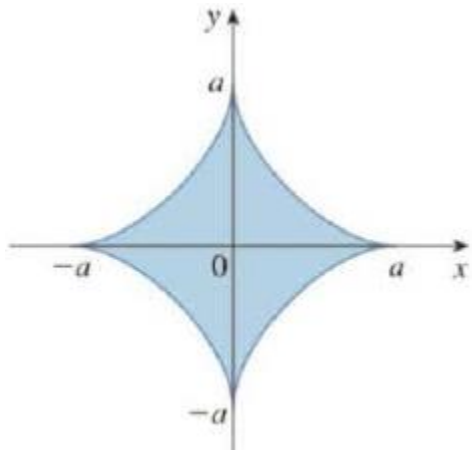
Illustration

Show that the area of asteroid $x^{2/3} + y^{2/3} = a^{2/3}$ is $\frac{3}{8}\pi a^2$.

Solution

The equation of asteroid is $x^{2/3} + y^{2/3} = a^{2/3}$. The graph is shown in the following figure. It is symmetrical about both the axes.

To find the area of the whole asteroid, first we find the area of the portion that lies in the first quadrant and then multiply by 4.



Let A be the area of the portion that lies in the first quadrant then

$$A = \int_0^a y\,dx$$

$$= \int_0^a \left(a^{2/3} - x^{2/3}\right)^{3/2} dx$$

Putting $x = a\sin^3 t$ then $dx = 3a\sin^2 t\cos t\,dt$. Here t varies from 0 to $\pi/2$.

Thus,

$$A = \int_0^a \left(a^{2/3} - a^{2/3}\sin^2 t\right)^{3/2} \times 3a\sin^2 t\cos t\,dt$$

$$= \int_0^{\pi/2} a\cos^3 t \times 3a\sin^2 t\cos t\,dt$$

$$= 3a^2 \int_0^{\pi/2} \sin^2 t\cos^4 t\,dt$$

$$= 3a^2 \times \frac{\Gamma\left(\frac{2+1}{2}\right)\Gamma\left(\frac{4+1}{2}\right)}{2\Gamma\left(\frac{2+4+2}{2}\right)} \qquad \left[\because \int_0^{\pi/2} \sin^m x\cos^n x\,dx = \frac{\Gamma\left(\frac{m+1}{2}\right)\Gamma\left(\frac{n+1}{2}\right)}{2\Gamma\left(\frac{m+n+2}{2}\right)}\right]$$

$$= 3a^2 \times \frac{\Gamma\left(\frac{3}{2}\right)\Gamma\left(\frac{5}{2}\right)}{2\Gamma(4)}$$

$$= 3a^2 \times \frac{\frac{1}{2}\times\Gamma\left(\frac{1}{2}\right)\times\frac{3}{2}\times\frac{1}{2}\times\Gamma\left(\frac{1}{2}\right)}{2(4-1)!} \qquad \left[\because \Gamma(n+1) = n\Gamma(n),\ \Gamma(n) = (n-1)!\right]$$

$$= 3a^2 \times \frac{\frac{1}{2}\times\sqrt{\pi}\times\frac{3}{2}\times\frac{1}{2}\times\sqrt{\pi}}{2\times3\times2\times1} \qquad \left[\because \Gamma\frac{1}{2} = \sqrt{\pi}\right]$$

$$= \frac{3}{32}\pi a^2$$

Thus, the area of the whole asteroid = $4 \times \frac{3}{32}\pi a^2 = \frac{3}{8}\pi a^2$.

Exercise for Reader

1. Find the area bounded by the curve $y = x^2 - 4$, x-axis and the ordinates at $x = 3$ and $x = 5$.
2. Find the area bounded by the curve $y = e^x$, x-axis and the ordinates at $x = a$ and $x = b$.
3. Compute the area of the region bounded by the curve $y^2 = 4x$ and the straight line $y = x$.
4. Compute the area of the region bounded by the curve $y^2 = 12x$ and the straight line $x = 12$.
5. Compute the area of the region bounded by the curves $y = \sqrt{x}$ and $y = x^2$.