Lecture 19

Learning Objectives

At the end of this class, students should be able to:

- understand the concept of definite integral
- use the properties of definite integral
- solve related problems

Definite Integrals

The definite integral of a continuous function $f(x)$ over the interval from $x = a$ to $x = b$ is denoted by $\int_a^b f(x)\,dx$ and described as the definite integral of $f(x)$ between the limits a and b, where a is called the lower limit and b the upper limit. The following theorem provides the key result for evaluating the definite integral.

Fundamental Theorem of Integral Calculus

Suppose that $f(x)$ is continuous on the interval $a \leq x \leq b$, and let $F(x)$ be an antiderivative of $f(x)$. Then

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

This theorem gives the connection between the derivative and the antiderivative of the function.

Properties of Definite Integrals

Assume that $f(x)$ and $g(x)$ are continuous function on the indicated intervals and k is a constant. Then

1. The value of the definite integral depends only upon the limits not on the variable of integration, i.e., $\int_a^b f(x)dx = \int_a^b f(z)dz$.
2. We can factor out a constant, i.e., $\int_a^b kf(x)dx = k\int_a^b f(x)dx$.
3. We can break up definite integrals across a sum or difference, i.e., $\int_a^b [f(x) \pm g(x)]dx = \int_a^b f(x)dx \pm \int_a^b g(x)dx$
4. An interchange of limits changes the sign of integral, i.e., $\int_a^b f(x)dx = -\int_b^a f(x)dx$.
5. $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$, where $a \leq c \leq b$.
6. If the lower limit of a definite integral is zero, then the variable in the integrand can be replaced by upper limit minus variable, i.e., $\int_0^a f(x)dx = \int_0^a f(a-x)dx$.

7. If $f(x)$ is an even function, then $\int_{-a}^{a} f(x)dx = 2\int_{0}^{a} f(x)dx$.
8. If $f(x)$ is an odd function, then $\int_{-a}^{a} f(x)dx = 0$.
9. If the upper and lower limits are the same then $\int_{a}^{a} f(x)dx = 0$.

Illustration

Evaluate: $\int_{2}^{5} 4xdx$

Solution

$\int_{2}^{5} 4xdx \quad = 4\int_{2}^{5} xdx$

$= \left.4\left(\frac{x^2}{2}\right)\right|_{2}^{5} = 4\left(\frac{5^2}{2} - \frac{2^2}{2}\right) = 4 \times \frac{21}{2} = 42$

Illustration

Evaluate: $\int_{0}^{2} \frac{4x}{\sqrt{x^2+1}}dx$

Solution

Let $x^2 + 1 = u$. Then

$2x\,dx = du$

Since this is a definite integral, the limit should be changed when the variable is changed.

When $x = 0$, $u = 0 + 1 = 1$; and when $x = 2$, $u = 2^2 + 1 = 5$.

$\therefore \int_{0}^{2} \frac{4x}{\sqrt{x^2+1}}dx \quad = \int_{1}^{5} \frac{2du}{\sqrt{u}}$

$= 2\int_{1}^{5} u^{-1/2}du$

$= \left.2\left(\frac{u^{1/2}}{1/2}\right)\right|_{1}^{5} = \left.4\left(u^{1/2}\right)\right|_{1}^{5}$

$= 4(5^{1/2} - 1^{1/2}) = 4(\sqrt{5} - 1)$

Illustration

Evaluate: $\int_{0}^{1} x^2e^x\,dx$

Solution

First solving it by removing the limits, so

$\int x^2e^xdx \quad = x^2\int e^xdx - \int\left\{\frac{d}{dx}(x^2)\int e^x\,dx\right\}dx$

$= x^2 \times e^x - \int\{2x \times e^x\}dx$

$$= x^2e^x - 2\int xe^x dx$$

$$= x^2e^x - 2\left[x\int e^x dx - \int\left\{\frac{d}{dx}(x)\int e^x\,dx\right\}dx\right]$$

$$= x^2e^x - 2[x \times e^x - \int\{1 \times e^x\}dx]$$

$$= x^2e^x - 2[xe^x - \int e^x dx]$$

$$= x^2e^x - 2[xe^x - e^x] + c$$

$$= e^x(x^2 - 2x + 2) + c$$

Now

$$\int_0^1 x^2e^x\,dx = \{e^x(x^2 - 2x + 2) + c\}|_0^1$$
$$= [e^1(1^2 - 2 \times 1 + 2) + c] - [e^0(0^2 - 2 \times 0 + 2) + c]$$
$$= e - 2$$

Illustration

Evaluate: $\int_0^{\pi/2} \frac{\sin x}{\sin x + \cos x} dx$

Solution

Let, $I = \int_0^{\pi/2} \frac{\sin x}{\sin x + \cos x} dx$ (i)

$= \int_0^{\pi/2} \frac{\sin(\frac{\pi}{2} - x)}{\sin(\frac{\pi}{2} - x) + \cos(\frac{\pi}{2} - x)} dx$ $\left[\because \int_0^a f(x)dx = \int_0^a f(a - x)dx\right]$

$= \int_0^{\pi/2} \frac{\cos x}{\cos x + \sin x} dx$ (ii)

Adding equation (i) and (ii), we get

$2I = \int_0^{\pi/2} \frac{\sin x + \cos x}{\sin x + \cos x} dx$

or, $2I = \int_0^{\pi/2} dx$

or, $2I = [x]_0^{\pi/2} = \pi/2$

∴ $I = \pi/4$

Illustration

Evaluate $\int_0^a \frac{\sqrt{x}}{\sqrt{x} + \sqrt{a - x}} dx$

Solution

Let, $I = \int_0^a \frac{\sqrt{x}}{\sqrt{x} + \sqrt{a - x}} dx$ (i)

$= \int_0^a \frac{\sqrt{a - x}}{\sqrt{a - x} + \sqrt{x}} dx$ (ii) $\left[\because \int_0^a f(x)dx = \int_0^a f(a - x)dx\right]$

Adding equation (i) and (ii), we get

$$2I = \int_0^a \frac{\sqrt{x}+\sqrt{a-x}}{\sqrt{x}+\sqrt{a-x}}\,dx$$

or, $2I = \int_0^a dx$

or, $2I = [x]_0^a = a$

∴ $I = a/2$

Illustration

Evaluate: $\int_0^1 \frac{\ln(1+x)}{1+x^2}\,dx$

Solution

Let, $I = \int_0^1 \frac{\ln(1+x)}{1+x^2}\,dx$

putting $x = \tan t \quad \Rightarrow dx = \sec^2 t\,dt$

Thus, $I = \int_0^{\pi/4} \frac{\ln(1+\tan t)}{1+\tan^2 t} \sec^2 t\,dt$

$= \int_0^{\pi/4} \ln(1+\tan t)\,dt$

$= \int_0^{\pi/4} \ln\left[1+\tan(\frac{\pi}{4}-t)\right]dt$ $\quad\left[\because\ \int_0^a f(x)dx = \int_0^a f(a-x)dx\right]$

$= \int_0^{\pi/4} \ln\left[1+\frac{1-\tan t}{1+\tan t}\right]dt$

$= \int_0^{\pi/4} \ln\left[\frac{2}{1+\tan t}\right]dt$

$= \int_0^{\pi/4} [\ln 2 - \ln(1+\tan t)]\,dt$

$= \int_0^{\pi/4} \ln 2\,dt - \int_0^{\pi/4} \ln(1+\tan t)\,dt$

or, $I = \int_0^{\pi/4} \ln 2\,dt - I$ $\quad[\because I = \int_0^{\pi/4} \ln(1+\tan t)\,dt]$

or, $2I = \ln 2 \int_0^{\pi/4} dt$

$= \ln 2\,[t]_0^{\pi/4} = \frac{\pi}{4}\ln 2$

∴ $I = \frac{\pi}{8}\ln 2$

Illustration

Evaluate: $\int_0^{\pi/2} \frac{x\sin x\cos x}{\cos^4 x+\sin^4 x}\,dx$

Solution

Let, $I = \int_0^{\pi/2} \frac{x \sin x \cos x}{\cos^4 x + \sin^4 x} dx$ (i)

$= \int_0^{\pi/2} \frac{\left(\frac{\pi}{2} - x\right) \sin\left(\frac{\pi}{2} - x\right) \cos\left(\frac{\pi}{2} - x\right)}{\cos^4\left(\frac{\pi}{2} - x\right) + \sin^4\left(\frac{\pi}{2} - x\right)} dx$ $\left[\because \int_0^a f(x)dx = \int_0^a f(a-x)dx\right]$

$= \int_0^{\pi/2} \frac{\left(\frac{\pi}{2} - x\right) \cos x \sin x}{\sin^4 x + \cos^4 x} dx$

or, $I = \frac{\pi}{2} \int_0^{\pi/2} \frac{\sin x \cos x}{\cos^4 x + \sin^4 x} dx - \int_0^{\pi/2} \frac{x \sin x \cos x}{\cos^4 x + \sin^4 x} dx$

or, $I = \frac{\pi}{2} \int_0^{\pi/2} \frac{\sin x \cos x}{\cos^4 x + \sin^4 x} dx - I$

or, $2I = \frac{\pi}{2} \int_0^{\pi/2} \frac{\sin x \cos x}{\cos^4 x + \sin^4 x} dx$

Putting $\cos^2 x = t \Rightarrow -2 \sin x \cos x \, dx = dy$, we get

$= -\frac{\pi}{4} \int_1^0 \frac{1}{(1-t)^2 + t^2} dt$

$= \frac{\pi}{4} \int_0^1 \frac{1}{2t^2 - 2t + 1} dt$

$= \frac{\pi}{8} \int_0^1 \frac{1}{(t - \frac{1}{2})^2 + (\frac{1}{2})^2} dt$

$= \frac{\pi}{8} \times 2 \tan^{-1}(2t-1)\Big|_0^1$

$= \frac{\pi}{4} [\tan^{-1}(1) - \tan^{-1}(-1)]$

$= \frac{\pi}{4} \left[\frac{\pi}{4} - \left(-\frac{\pi}{4}\right)\right] = \frac{\pi^2}{8}$

$\therefore$ $I = \frac{\pi^2}{16}$

Exercise for Reader

Evaluate the following integrals.

1. $\int_0^1 3x^2 e^{x^3} dx$
2. $\int_1^3 \frac{2x+3}{x^2+3x} dx$
3. $\int_1^e \frac{(ln\, x)^2}{x} dx$
4. $\int_e^{e^2} \frac{1}{x\, ln\, x} dx$
5. $\int_0^1 x e^x dx$
6. $\int_1^2 x^2\, ln\, x\, dx$
7. $\int_0^a \frac{1}{x+\sqrt{a^2-x^2}} dx$
8. $\int_0^{\pi/2} \frac{\sqrt{\sin x}}{\sqrt{\sin x}+\sqrt{\cos x}} dx$
9. $\int_0^{\pi/2} \frac{x \tan x}{\sec x+\tan x} dx$
10. $\int_0^{\pi/2} \ln(\tan x)\, dx$