Lecture 18

Learning Objectives

At the end of this class, students should be able to:

- derive and use standard integrals
- evaluate the integral by partial fractions
- solve related problems

Integral of the form $\int e^x[f(x) + f'(x)]dx$

$\int e^x[f(x) + f'(x)]dx = e^x f(x) + c$, where $f(x)$ is a differentiable function of x.

We have $\int e^x[f(x) + f'(x)]dx$

$$= \int e^x f(x)dx + \int e^x f'(x)dx$$

$$= \int f(x)e^x dx + \int f'(x)e^x dx$$

Applying the rule of Integrating by Parts to the integral $\int f(x)e^x dx$, we get

$$= f(x)e^x - \int f'(x)e^x dx + \int f'(x)e^x dx + c$$

$$= f(x)e^x + c$$

Which is the right-hand side.

Illustration

Evaluate: $\int(5 + \tan x + \sec^2 x)e^x dx$

Solution

We have, $\int e^x[(5 + \tan x) + \sec^2 x]dx$

Let $f(x) = 5 + \tan x$ then $f'(x) = \sec^2 x$

Thus, $\int e^x[(5 + \tan x) + \sec^2 x]dx \quad = (5 + \tan x)\, e^x + c$

$$[\because \int e^x[f(x) + f'(x)]dx = e^x f(x) + c]$$

Illustration

Evaluate: $\int e^{\sin x}(\sin x \cos x + \cos x)dx$

Solution

We have, $\int e^{\sin x}(\sin x \cos x + \cos x)dx$

$$= \int e^{\sin x} \cos x\, (\sin x + 1)dx$$

Let $\sin x = t$ then $\cos x\, dx = dt$

$$= \int e^t(t + 1)dt$$

Let $f(t) = t$ then $f'(t) = 1$

Thus, $\int e^t(t+1)dt = te^t + c$ $\quad [\because \int e^x[f(x) + f'(x)]dx = e^x f(x) + c]$

$= \sin x\, e^{\sin x} + c$ $\quad [\because \sin x = t]$

Integrals of $\int \sqrt{a^2 - x^2}\, dx$, $\int \sqrt{x^2 - a^2}\, dx$, $\int \sqrt{x^2 + a^2}\, dx$

We can use substitution techniques to evaluate above integrals. However, the method of integration by parts is comparatively simpler and less time consuming.

a) $\int \sqrt{x^2 + a^2}\, dx = \frac{x}{2}\sqrt{x^2 + a^2} + \frac{a^2}{2}\ln\left[x + \sqrt{x^2 + a^2}\right] + c$

Let $I = \int \sqrt{x^2 + a^2}\, dx$

$= \int \sqrt{x^2 + a^2}\,(1)dx$

$= \sqrt{x^2 + a^2}.x - \int \frac{x}{\sqrt{x^2+a^2}}.x\, dx$ $\quad \left[\because \frac{d}{dx}\left(\sqrt{x^2 + a^2}\right) = \frac{1}{2\sqrt{x^2+a^2}}.2x = \frac{x}{\sqrt{x^2+a^2}}\right]$

$= x\sqrt{x^2 + a^2} - \int \frac{x^2}{\sqrt{x^2+a^2}}\, dx$

$= x\sqrt{x^2 + a^2} - \int \frac{(x^2+a^2)-a^2}{\sqrt{x^2+a^2}}\, dx$

$= x\sqrt{x^2 + a^2} - \int \sqrt{x^2 + a^2}\, dx + a^2 \int \frac{1}{\sqrt{x^2+a^2}}\, dx$

or, $I = x\sqrt{x^2 + a^2} - I + a^2 \ln\left[x + \sqrt{x^2 + a^2}\right] + k$ $\quad \left[\because \int \frac{1}{\sqrt{x^2+a^2}}dx = \ln\left[x + \sqrt{x^2 + a^2}\right] + c\right]$

or, $2I = x\sqrt{x^2 + a^2} + a^2 \ln\left[x + \sqrt{x^2 + a^2}\right] + k$

$\therefore I = \frac{x}{2}\sqrt{x^2 + a^2} + \frac{a^2}{2}\ln\left[x + \sqrt{x^2 + a^2}\right] + c$

b) $\int \sqrt{x^2 - a^2}\, dx = \frac{x}{2}\sqrt{x^2 - a^2} - \frac{a^2}{2}\ln\left[x + \sqrt{x^2 - a^2}\right] + c$

c) $\int \sqrt{a^2 - x^2}\, dx = \frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a} + c$

Integrals of the form $\int \sqrt{ax^2 + bx + c}\, dx$

we express the quadratic expression $ax^2 + bx + c$ as the sum or difference of two squares. Besides, we can also evaluate those integrals that can be reduced to this form.

Illustration

Evaluate: $\int \sqrt{2x^2 + 3x + 4}\, dx$

Solution

Here, $2x^2 + 3x + 4 = 2\left(x^2 + \frac{3}{2}x + 2\right)$

$$= 2\left(x^2 + 2 \times \frac{3}{4} \times x + \frac{9}{16} - \frac{9}{16} + 2\right)$$

$$= 2\left[\left(x + \frac{3}{4}\right)^2 - \left(\frac{\sqrt{23}}{2}\right)^2\right]$$

Thus,

$$\int \sqrt{2x^2 + 3x + 4}\, dx = \sqrt{2} \int \sqrt{\left(x + \frac{3}{4}\right)^2 - \left(\frac{\sqrt{23}}{2}\right)^2}\, dx$$

$$= \sqrt{2}\left[\frac{1}{2}\left(x + \frac{3}{4}\right)\sqrt{\left(x + \frac{3}{4}\right)^2 - \left(\frac{\sqrt{23}}{2}\right)^2} - \frac{1}{2}\left(\frac{\sqrt{23}}{2}\right)^2 \ln\left\{\left(x + \frac{3}{4}\right) + \sqrt{\left(x + \frac{3}{4}\right)^2 - \left(\frac{\sqrt{23}}{2}\right)^2}\right\}\right] + c$$

$$\left[\because \int \sqrt{x^2 - a^2}\, dx = \frac{x}{2}\sqrt{x^2 - a^2} - \frac{a^2}{2}\ln\left[x + \sqrt{x^2 - a^2}\right] + c\right]$$

$$= \sqrt{2}\left[\frac{1}{2}\left(\frac{4x+3}{4}\right)\sqrt{\frac{2x^2+3x+4}{2}} - \frac{1}{2} \times \frac{23}{4}\ln\left\{\left(x + \frac{3}{4}\right) + \sqrt{\left(x^2 + \frac{3}{2}x + 2\right)}\right\}\right] + c$$

$$= \frac{1}{8}(4x + 3)\sqrt{2x^2 + 3x + 4} - \frac{23}{4\sqrt{2}}\ln\left\{\left(x + \frac{3}{4}\right) + \sqrt{\left(x^2 + \frac{3}{2}x + 2\right)}\right\} + c$$

Integral of the form $\int (px + q)\sqrt{ax^2 + bx + c}\, dx$

In this case, we find two constants *A* and *B* such that

$$px + q = A\frac{d}{dx}(ax^2 + bx + c) + B$$

i.e., $px + q = A(2ax + b) + B$

Now, by equating coefficients of *x* and the constants on both sides, we find the values of *A* and *B*.

Thus,

$$\int (px + q)\sqrt{ax^2 + bx + c}\, dx = A\int (2ax + b)\sqrt{ax^2 + bx + c}\, dx + B\int \sqrt{ax^2 + bx + c}\, dx$$

Illustration

Evaluate: $\int (x - 1)\sqrt{5 - 4x - x^2}\, dx$

Solution

Let $x - 1 = A\frac{d}{dx}(5 - 4x - x^2) + B$

i.e., $x - 1 = A(-4 - 2x) + B$

Equating coefficients of *x* and the constants on both sides, we get

$-2A = 1 \quad \Rightarrow A = -1/2$

and $-4A + B = -1 \quad \Rightarrow B = -3$

Thus, $\int (x-1)\sqrt{5-4x-x^2}\,dx \qquad = -\frac{1}{2}\int(-4-2x)\sqrt{5-4x-x^2}\,dx - 3\int\sqrt{5-4x-x^2}\,dx$

Substitute $5-4x-x^2 = t$ in the first integral. Then $(-4x-2x)dx = dt$

$$= -\frac{1}{2}\int t^{\frac{1}{2}}dt - 3\int\sqrt{3^2-(x+2)^2}\,dx$$

$$[\because\ 5-4x-x^2 = -(x^2+4x-5) = -(x^2+2\times2\times x+4-9) = -\{(x+2)^2-3^2\}]$$

$$= -\frac{1}{2}\times\frac{2}{3}t^{\frac{3}{2}} - 3\left[\frac{(x+2)}{2}\sqrt{3^2-(x+2)^2}+\frac{3^2}{2}\sin^{-1}\left(\frac{x+2}{3}\right)\right]+c$$

$$\left[\because\ \int\sqrt{a^2-x^2}\,dx = \frac{x}{2}\sqrt{a^2-x^2}+\frac{a^2}{2}\sin^{-1}\frac{x}{a}+c\right]$$

$$= -\frac{1}{3}(5-4x-x^2)^{\frac{3}{2}} - \frac{3}{2}(x+2)\sqrt{5-4x-x^2} - \frac{27}{2}\sin^{-1}\left(\frac{x+2}{3}\right)+c$$

Integration by Partial Fractions

A group of fractions connected by the signs addition and subtraction is reduced to a simpler form, i.e., a single fraction whose denominator is L. C. M. of the given fractions. For example,

$$\frac{1}{x+1}-\frac{1}{x+2} = \frac{(x+2)-(x+1)}{(x+1)(x+2)} = \frac{1}{(x+1)(x+2)}$$

The reverse process of separating a fraction into group of simple fractions is known as partial fractions. Let us talk about various cases.

Case I

When the numerator is of lower degree than the denominator. Here two cases arise the linear factors in the denominator may be **a)** repeated and **b)** non-repeated. When we have non-repeated linear factors in the denominator, we write the given fraction in the following form and find the values of A and B.

$$\frac{p(x)}{(ax+b)(cx+d)} = \frac{A}{(ax+b)}+\frac{B}{(cx+d)}$$

For example, the expression $\frac{7x}{x^2-3x+2}$ which is equivalent to $\frac{7x}{(x-1)(x-2)}$ can be written as:

$$\frac{7x}{(x-1)(x-2)} = \frac{A}{x-1}+\frac{B}{x-2}$$

or, $\quad 7x = A(x-2)+B(x-1).$

To find A, put $x = 1$, we get,

$7\times(1) = A(1-2)+B(1-1)$, i.e., $A = -7$.

Similarly, to find B, put $x = 2$, we get,

$7\times(2) = A(2-2)+B(2-1)$, i.e., $B = 14$.

$\therefore \qquad \frac{7x}{(x-1)(x-2)} = \frac{-7}{(x-1)} + \frac{14}{(x-2)}$

When some linear factors of the denominator of the given fraction are repeated, then

$$\frac{p(x)}{(ax+b)(cx+d)^2} = \frac{A}{(ax+b)} + \frac{B}{(cx+d)} + \frac{C}{(cx+d)^2}$$

Case II

When one of the factors in the denominator is quadratic, then

$$\frac{p(x)}{(ax+b)(cx^2+dx+e)} = \frac{A}{(ax+b)} + \frac{Bx+C}{(cx^2+dx+e)}$$

Case III

When the quadratic factor in the denominator is repeated, then

$$\frac{p(x)}{(ax+b)(cx^2+dx+e)^2} = \frac{A}{(ax+b)} + \frac{Bx+C}{(cx^2+dx+e)} + \frac{Dx+E}{(cx^2+dx+e)^2}$$

Illustration

Evaluate $\int \frac{3x}{x^2+5x+6} dx$

Solution

Here, Denominator = $x^2 + 5x + 6 = (x+2)(x+3)$

$\therefore \qquad \frac{3x}{x^2+5x+6} = \frac{3x}{(x+2)(x+3)}$

Let $\qquad \frac{3x}{(x+2)(x+5)} = \frac{A}{x+2} + \frac{B}{x+3}$

or $\qquad \frac{3x}{(x+2)(x+5)} = \frac{A(x+3)+B(x+2)}{(x+2)(x+3)}$

Equating both sides, we get

$$3x = A(x+3) + B(x+2)$$

Putting x = -3, we get -9 = -B, or B = 9.

Again, putting x = -2, we get A = -6.

$\therefore \qquad \frac{3x}{(x+2)(x+5)} = \frac{-6}{x+2} + \frac{9}{x+3}$

Now, $\int \frac{3x}{x^2+5x+6} dx = \int \left(-\frac{6}{x+2} + \frac{9}{x+3}\right) dx = -6\, ln(x+2) + 9\, ln(x+3) + c$

Exercise for Reader

Evaluate the following integrals.

1. $\int e^x(\sin x + \cos x)dx$
2. $\int e^x(\cot x + \ln(\sin x))dx$
3. $\int \sqrt{\frac{x+1}{x+3}}$ [Hint: multiply numerator and denominator by $\sqrt{x+1}$.
4. $\int \sqrt{2x^2 + 2x + 5}\ dx$
5. $\int \sqrt{3x^2 + 4x + 1}\ dx$
6. $\int (x + 3)\sqrt{5 - 4x - x^2}\ dx$
7. $\int \frac{x^2-4}{(x^2+1)(x^2+3)} dx$
8. $\int \frac{1}{x[6(\ln x)^2+7\ln x+2]} dx$