Lecture 17

Learning Objectives

At the end of this class, students should be able to:

- evaluate the integral by using integration by parts
- solve related problems

Integration by Parts

Let u and v be two functions of x. then

$$\int \boldsymbol{uvdx} = \boldsymbol{u}\int \boldsymbol{vdx} - \int \left\{\frac{\boldsymbol{d}}{\boldsymbol{dx}}(\boldsymbol{u})\int \boldsymbol{vdx}\right\}\boldsymbol{dx}$$

This is the integral of the product of two functions, and is known as integration by parts. We can express the result as under:

$\int uvdx$ = (1^{st} function) × (Integral of 2^{nd} function)
– Integral of {(derivative of 1^{st} function) × (Integral of 2^{nd} function)}

where, u: 1^{st} function and v: 2^{nd} function

How can we select u and v to make integration by parts work? As a general guideline, we do the following.

First, identify the types of functions occurring in the problem in the order: "I-L-A-T-E"
(There are always exceptions, but these are generally helpful.)

I: Inverse Trigonometric Function

L: Logarithmic Function

A: Algebraic Function

T: Trigonometric Function

E: Exponential Function

Second, choose u to equal the function whose type occurs first on the list. Then v equals the rest of the integrand so that uv equals the original integrand.

Let us try to understand with the help of the following illustrations.

Illustration

Evaluate $\int x\, ln\, x\, dx$

Solution

The integral contains a logarithmic function ($ln\, x$) and an algebraic function (x). Thus, let $u = ln\, x$ and $v = x$. Then the given integral should be written as $\int (ln\, x) x dx$.

Now

$$\int (ln\, x) x dx = ln\, x \int x dx - \int \left\{ \frac{d}{dx}(ln\, x) \int x\, dx \right\} dx$$

$$= ln\, x \times \frac{x^2}{2} - \int \left\{ \frac{1}{x} \times \frac{x^2}{2} \right\} dx$$

$$= \frac{x^2}{2} ln\, x - \frac{1}{2} \int x dx$$

$$= \frac{x^2}{2} ln\, x - \frac{1}{2} \times \frac{x^2}{2} + c$$

$$= \frac{x^2}{2} ln\, x - \frac{x^2}{4} + c$$

Illustration

Evaluate $\int x e^x dx$

Solution

The integral contains an algebraic (x) and exponential (e^x). Thus, let $u = x$ and $v = e^x$.

Now

$$\int x e^x dx = x \int e^x dx - \int \left\{ \frac{d}{dx}(x) \int e^x\, dx \right\} dx$$

$$= x \times e^x - \int \{1 \times e^x\} dx$$

$$= x e^x - \int e^x dx$$

$$= x e^x - e^x + c$$

$$= e^x (x - 1) + c$$

Illustration

Evaluate $\int x^2 \sin x\, dx$

Solution

Since x^2 is an algebraic function and $\sin x$ is a trigonometric function. Thus, let $u = x^2$ and $v = \sin x$.

Now,

$$\int x^2 \sin x\, dx = x^2 \int \sin x\, dx - \int \{ \frac{d}{dx}(x^2) \int \sin x\, dx \}\, dx$$

$$= x^2(-\cos x) - \int \{2x \times (-\cos x)\} dx$$

$$= -x^2 \cos x + 2 \int x \cos x\, dx$$

$$= -x^2 \cos x + 2[x \int \cos x\, dx - \int \left\{ \frac{d}{dx}(x) \int \cos x\, dx \right\} dx]$$

$$= -x^2 \cos x + 2[x \sin x - \int \sin x\, dx]$$

$= -x^2 \cos x + 2[x \ \sin x + \cos x] + c$

$= -x^2 \cos x + 2x \ \sin x + 2\cos x + c$

In this example, integration by parts technique was adopted two times to obtain an answer. We should not switch choices for *u* and *v* in successive applications.

Illustration

Evaluate $\int x\sqrt{x+5}dx$

Solution

Since both of these are algebraic functions, the ILATE Rule of Thumb is not helpful. In such cases, we take *v* as the most complicated portion of the integrand that can be "easily' integrated and *u* be that portion of the integrand whose derivative is a "simpler" function than *u* itself.

Thus, let $v = \sqrt{x+5}$ and $u = x$.

Now

$$\int x\sqrt{x+5}dx = x\int \sqrt{x+5}dx - \int \left\{\frac{d}{dx}(x)\int \sqrt{x+5}\,dx\right\}dx$$

$$= x \times \frac{2}{3}(x+5)^{3/2} - \int \left\{1 \times \frac{2}{3}(x+5)^{3/2}\right\}dx$$

$$= \frac{2x}{3}(x+5)^{3/2} - \frac{2}{3}\int (x+5)^{3/2}dx$$

$$= \frac{2x}{3}(x+5)^{3/2} - \frac{2}{3} \times \frac{2}{5}(x+5)^{5/2} + c$$

$$= \frac{2x}{3}(x+5)^{3/2} - \frac{4}{15}(x+5)^{5/2} + c$$

Illustration

Evaluate $\int \frac{x\tan^{-1}x}{(1+x^2)^{3/2}}dx$

Solution

Let $\tan^{-1}x = u \ \Rightarrow \tan u = x$ and $\frac{1}{1+x^2}dx = du$

Thus, $\int \frac{x\tan^{-1}x}{(1+x^2)^{3/2}}dx \quad = \int \frac{\tan u \times u}{(1+\tan^2 u)^{1/2}}du$

$$= \int \frac{u\tan u}{\sec u}du$$

$$= \int u \sin u \, du$$

$$= u(-\cos u) + \int \cos u \, du$$

$$= -u\cos u + \sin u + c$$

$$= -\tan^{-1}x \times \frac{1}{\sqrt{1+x^2}} + \frac{x}{\sqrt{1+x^2}} + c \qquad [\because \tan u = x]$$

$$= (-\tan^{-1} x + x)\frac{1}{\sqrt{1+x^2}} + c$$

Tabular Integration

The tabular method can be applied to any function which is the product of two expressions (e.g., $\int uvdx$), where one of the expressions has some nth derivative equal to zero.

The tabular method uses a convenient table with three columns. We can call the first column "Signs," the second column "u" and the third column "dv." Under the column called "Signs," we list positive and negative signs in alternating order for as many times as the problem requires. The first entry in the column labeled "u" will be the part of the function we want to integrate which can be reduced to zero through successive differentiation. The first entry in the column labeled "dv" includes the other part of the function we want to integrate. The subsequent entries in this column will be the successive integrals of the first entry.

Let us try to understand the procedure with the help of following illustration.

Illustration

Evaluate $\int x^5 \sin x\, dx$

Solution

let $u = x^5$ and $v = \sin x$. The table for finding $\int x^5 \sin x\, dx$ looks as follows:

Signs	u	dv
+	x^5	$\sin x$
$-$	$5x^4$	$-\cos x$
+	$20x^3$	$-\sin x$
$-$	$60x^2$	$\cos x$
+	$120x$	$\sin x$
$-$	120	$-\cos x$
+	0	$-\sin x$

Using this table to find the indefinite integral of the function requires taking the sign from each row in the column except the last, applying it to the entry for "u" in the same row, and multiplying the result by the entry for "dv" in the next row and add them together to get the final indefinite integral.

$\therefore \int x^5 \sin x \, dx$

$$= -x^5 \cos x + 5x^4 \sin x + 20x^3 \cos x - 60x^2 \sin x - 120x \cos x + 120 \sin x + c$$

Exercise for Reader

Evaluate the following integrals.

1. $\int xe^{-x} dx$
2. $\int x^2 e^x dx$
3. $\int (x+1)e^{2x}$
4. $\int x^4 e^{5x} dx$
5. $\int x^2 \sqrt{1+x} \, dx$
6. $\int x^2 \ln x \, dx$
7. $\int x \sin^2 x \; dx$
8. $\int \sin^{-1}\left(\frac{2x}{1+x^2}\right) dx$
9. $\int \tan^{-1} \sqrt{\frac{1-x}{1+x}} \, dx$