

## Lecture 16

### Learning Objectives

At the end of this class, students should be able to:

- derive and use standard integrals
- solve related problems

### Trigonometric substitutions

- If the integrand involves  $\sqrt{a^2 - x^2}$ , then we can put  $x = a \cos t$  or  $y = a \sin t$ .
- If the integrand involves  $\sqrt{a^2 + x^2}$  or  $a^2 + x^2$ , then we can put  $x = a \tan t$ .
- If the integrand involves  $\sqrt{x^2 - a^2}$ , then we can put  $x = a \sec t$ .

The idea behind these substitutions is to get rid of the square root sign by using trigonometric identities:  $1 - \cos^2 t = \sin^2 t$ ,  $1 + \tan^2 t = \sec^2 t$  and  $\sec^2 t - 1 = \tan^2 t$ .

$$\text{a) } \int \frac{1}{\sqrt{a^2 - x^2}} dx = \sin^{-1} \left( \frac{x}{a} \right) + c$$

Let  $x = a \sin t$  then  $dx = a \cos t dt$

$$\begin{aligned} \text{Thus, } \int \frac{1}{\sqrt{a^2 - x^2}} dx &= \int \frac{1}{\sqrt{a^2 - a^2 \sin^2 t}} \times a \cos t dt \\ &= \int \frac{a \cos t}{a \cos t} dt \\ &= \int dt \\ &= t + c \\ &= \sin^{-1} \left( \frac{x}{a} \right) + c \quad [\because x = a \sin t] \end{aligned}$$

$$\text{b) } \int \frac{1}{x\sqrt{x^2 - a^2}} dx = \frac{1}{a} \sec^{-1} \left( \frac{x}{a} \right) + c$$

Let  $x = a \sec t$  then  $dx = a \sec t \tan t dt$

$$\begin{aligned} \text{Thus, } \int \frac{1}{x\sqrt{x^2 - a^2}} dx &= \int \frac{1}{a \sec t \sqrt{a^2 \sec^2 t - a^2}} \times a \sec t \tan t dt \\ &= \frac{1}{a} \int dt \\ &= \frac{1}{a} t + c \\ &= \frac{1}{a} \sec^{-1} \left( \frac{x}{a} \right) + c \end{aligned}$$

$$\text{c) } \int \frac{1}{\sqrt{x^2 - a^2}} dx = \ln|x + \sqrt{x^2 - a^2}| + c = \cosh^{-1}\left(\frac{x}{a}\right) + c$$

Let  $x = a \sec t$  then  $dx = a \sec t \tan t dt$

$$\begin{aligned} \text{Thus, } \int \frac{1}{\sqrt{x^2 - a^2}} dx &= \int \frac{1}{\sqrt{a^2 \sec^2 t - a^2}} \times a \sec t \tan t dt \\ &= \int \sec t dt \\ &= \ln|\sec t + \tan t| + k \\ &= \ln\left|\frac{x}{a} + \frac{\sqrt{x^2 - a^2}}{a}\right| + k \\ &= \ln|x + \sqrt{x^2 - a^2}| - \ln a + k \\ &= \ln|x + \sqrt{x^2 - a^2}| + c \end{aligned}$$

$$\text{d) } \int \frac{1}{\sqrt{x^2 + a^2}} dx = \ln|x + \sqrt{x^2 + a^2}| + c = \sinh^{-1}\left(\frac{x}{a}\right) + c$$

Let us try to apply these standard results in the following illustrations.

*Illustration*

Evaluate:  $\int \frac{dx}{\sqrt{5 + 4x - x^2}}$

*Solution*

$$\begin{aligned} \text{Here, } 5 + 4x - x^2 &= -(x^2 - 4x - 5) \\ &= -(x^2 - 2 \times 2 \times x + 4 - 9) \\ &= -[(x - 2)^2 - 3^2] = 3^2 - (x - 2)^2 \end{aligned}$$

$$\begin{aligned} \text{Thus, } \int \frac{dx}{\sqrt{5 + 4x - x^2}} &= \int \frac{dx}{\sqrt{3^2 - (x - 2)^2}} \\ &= \sin^{-1}\left(\frac{x - 2}{3}\right) + c \end{aligned}$$

*Illustration*

Evaluate:  $\int \frac{dx}{\sqrt{2x^2 + 3x + 4}}$

*Solution*

$$\begin{aligned} \text{Here, } 2x^2 + 3x + 4 &= 2\left(x^2 + \frac{3}{2}x + 2\right) \\ &= 2\left(x^2 + 2 \times \frac{3}{4} \times x + \frac{9}{16} - \frac{9}{16} + 2\right) \\ &= 2\left[\left(x + \frac{3}{4}\right)^2 - \left(\frac{\sqrt{23}}{2}\right)^2\right] \end{aligned}$$

$$\begin{aligned}
\text{Thus, } \int \frac{dx}{\sqrt{2x^2+3x+4}} &= \frac{1}{\sqrt{2}} \int \frac{dx}{\sqrt{\left(x+\frac{3}{4}\right)^2 - \left(\frac{\sqrt{23}}{2}\right)^2}} \\
&= \frac{1}{\sqrt{2}} \ln \left| \left(x + \frac{3}{4}\right) + \sqrt{\left(x + \frac{3}{4}\right)^2 - \left(\frac{\sqrt{23}}{2}\right)^2} \right| + c \\
&= \frac{1}{\sqrt{2}} \ln \left| \left(x + \frac{3}{4}\right) + \sqrt{x^2 + \frac{3}{2}x + 2} \right| + c
\end{aligned}$$

*Illustration*

Evaluate:  $\int \frac{2 \cos x}{\sqrt{4-\sin^2 x}} dx$

*Solution*

Let  $\sin x = t$  then  $\cos x dx = dt$

$$\begin{aligned}
\text{Thus, } \int \frac{2 \cos x}{\sqrt{4-\sin^2 x}} dx &= 2 \int \frac{dt}{\sqrt{2^2-t^2}} \\
&= 2 \sin^{-1} \left( \frac{t}{2} \right) + c \\
&= 2 \sin^{-1} \left( \frac{\sin x}{2} \right) + c
\end{aligned}$$

**Integrals of the form  $\int \frac{px+q}{\sqrt{ax^2+bx+c}} dx$  and  $\int \frac{px+q}{ax^2+bx+c} dx$**

Here, the numerator is a linear polynomial and denominator is a quadratics polynomial. In this case, we express the numerator in the form  $\left[ A \frac{d}{dx} (\text{denominator}) + B \right]$  where  $A$  and  $B$  are constants.

i.e.,  $px + q = A \frac{d}{dx} (ax^2 + bx + c) + B$

or,  $px + q = A(2ax + b) + B$

Now, by equating coefficients of  $x$  and the constants on both sides, we find the values of  $A$  and  $B$ .

Thus,  $\int \frac{px+q}{\sqrt{ax^2+bx+c}} dx = A \int \frac{2ax+b}{\sqrt{ax^2+bx+c}} dx + B \int \frac{1}{\sqrt{ax^2+bx+c}} dx$

and  $\int \frac{px+q}{ax^2+bx+c} dx = A \int \frac{2ax+b}{ax^2+bx+c} dx + B \int \frac{1}{ax^2+bx+c} dx$

Let us try to understand from the following illustration.

*Illustration*

Evaluate:  $\int \frac{2x+5}{\sqrt{2x^2+2x+5}} dx$

*Solution*

$$\text{Let } 2x + 5 = A \frac{d}{dx} (2x^2 + 2x + 5) + B$$

$$\text{i.e., } 2x + 5 = A(4x + 2) + B$$

Equating coefficients of  $x$  and the constants on both sides, we get

$$4A = 2 \quad \Rightarrow A = 1/2$$

$$\text{and } 2A + B = 5 \quad \Rightarrow B = 4$$

$$\text{Thus, } \int \frac{2x+5}{\sqrt{2x^2+2x+5}} dx = \frac{1}{2} \int \frac{4x+2}{\sqrt{2x^2+2x+5}} dx + \int \frac{4}{\sqrt{2x^2+2x+5}} dx$$

Substitute  $2x^2 + 2x + 5 = t$  in the first integral. Then  $(4x + 2)dx = dt$

$$= \frac{1}{2} \int \frac{dt}{t^{1/2}} + \frac{1}{\sqrt{2}} \int \frac{4}{\sqrt{\left(x+\frac{1}{2}\right)^2 + \left(\frac{3}{2}\right)^2}} dx$$

$$\left[ \because 2x^2 + 2x + 5 = 2 \left( x^2 + x + \frac{5}{2} \right) = 2 \left( x^2 + 2 \times \frac{1}{2} \times x + \frac{1}{4} - \frac{1}{4} + \frac{5}{2} \right) \right. \\ \left. = 2 \left\{ \left( x + \frac{1}{2} \right)^2 + \left( \frac{3}{2} \right)^2 \right\} \right]$$

$$= \frac{1}{2} \int t^{-1/2} dt + \frac{4}{\sqrt{2}} \ln \left[ \left( x + \frac{1}{2} \right) + \sqrt{\left( x + \frac{1}{2} \right)^2 + \left( \frac{3}{2} \right)^2} \right]$$

$$= \frac{1}{2} \left[ \frac{t^{1/2}}{1/2} \right] + \frac{4}{\sqrt{2}} \ln \left[ \left( x + \frac{1}{2} \right) + \sqrt{x^2 + x + \frac{5}{2}} \right] + c$$

$$= \sqrt{2x^2 + 2x + 5} + 2\sqrt{2} \ln \left[ \left( x + \frac{1}{2} \right) + \sqrt{x^2 + x + \frac{5}{2}} \right] + c$$

**Integral of the form  $\int \frac{ae^x+b}{ce^x+d} dx$**

In this case, we express the numerator in the form  $\left[ A(\text{denominator}) + B \frac{d}{dx}(\text{denominator}) \right]$  where  $A$  and  $B$  are constants. Let us try to understand from the following illustration.

*Illustration*

$$\text{Evaluate: } \int \frac{3e^x+7}{4e^x+5} dx$$

*Solution*

$$\text{Let } 3e^x + 7 = A(4e^x + 5) + B \frac{d}{dx} (4e^x + 5)$$

$$\text{or, } 3e^x + 7 = A(4e^x + 5) + B(4e^x)$$

Now, by equating coefficients of  $e^x$  and the constants on both sides, we find  $A = 7/5$  and  $B = -13/20$ .

$$\begin{aligned}
\text{Thus, } \int \frac{3e^{x+7}}{4e^{x+5}} dx &= \frac{7}{5} \int \frac{4e^{x+5}}{4e^{x+5}} dx - \frac{13}{20} \int \frac{4e^x}{4e^{x+5}} dx \\
&= \frac{7}{5} \int dx - \frac{13}{20} \int \frac{4e^x}{4e^{x+5}} dx \\
&= \frac{7}{5}x - \frac{13}{20} \ln(4e^x + 5) + c
\end{aligned}$$

### Integral of the form $\int \frac{a \sin x + b \cos x}{c \sin x + d \cos x} dx$

In this case, we express the numerator in the form  $\left[ A(\text{denominator}) + B \frac{d}{dx}(\text{denominator}) \right]$  where  $A$  and  $B$  are constants. Let us try to understand from the following illustration.

#### Illustration

Evaluate:  $\int \frac{3 \sin x + 4 \cos x}{4 \sin x + \cos x} dx$

#### Solution

Let  $3 \sin x + 4 \cos x = A(4 \sin x + \cos x) + B \frac{d}{dx}(4 \sin x + \cos x)$

or,  $3 \sin x + 4 \cos x = A(4 \sin x + \cos x) + B(4 \cos x - \sin x)$

Now, by equating coefficients of  $\sin x$  and  $\cos x$  on both sides, we find

$$4A - B = 3 \text{ and } A + 4B = 4$$

Solving above equations, we get  $A = 16/17$  and  $B = 13/17$ .

$$\begin{aligned}
\text{Thus, } \int \frac{3 \sin x + 4 \cos x}{4 \sin x + \cos x} dx &= \frac{16}{17} \int \frac{4 \sin x + \cos x}{4 \sin x + \cos x} dx + \frac{13}{17} \int \frac{4 \cos x - \sin x}{4 \sin x + \cos x} dx \\
&= \frac{16}{17} \int dx + \frac{13}{17} \int \frac{4 \cos x - \sin x}{4 \sin x + \cos x} dx \\
&= \frac{16}{17}x + \frac{13}{17} \ln(4 \sin x + \cos x) + c
\end{aligned}$$

### Integral of the form $\int \frac{px^2 + qx + r}{ax^2 + bx + c} dx$

In this case, we express the numerator in the form  $\left[ A(\text{denominator}) + B \frac{d}{dx}(\text{denominator}) + C \right]$  where  $A$ ,  $B$  and  $C$  are constants. Let us try to understand from the following illustration.

#### Illustration

Evaluate:  $\int \frac{2x^2 + 3x + 4}{x^2 + 6x + 10} dx$

#### Solution

Let  $2x^2 + 3x + 4 = A(x^2 + 6x + 10) + B \frac{d}{dx}(x^2 + 6x + 10) + C$

or,  $2x^2 + 3x + 4 = A(x^2 + 6x + 10) + B(2x + 6) + C$

Now, by equating coefficients of  $x^2$ ,  $x$  and constant terms on both sides, we get

$$A = 2, 6A + 2B = 3 \text{ and } 10A + 6B + C = 4$$

Solving above equations, we get  $A = 2$ ,  $B = -9/2$  and  $C = 11$ .

$$\begin{aligned} \text{Thus, } \int \frac{2x^2+3x+4}{x^2+6x+10} dx &= 2 \int \frac{x^2+6x+10}{x^2+6x+10} dx - \frac{9}{2} \int \frac{2x+6}{x^2+6x+10} dx + 11 \int \frac{1}{x^2+6x+10} dx \\ &= 2 \int dx - \frac{9}{2} \int \frac{2x+6}{x^2+6x+10} dx + 11 \int \frac{1}{(x+3)^2+1} dx \\ &= 2x - \frac{9}{2} \ln(x^2 + 6x + 10) + 11 \tan^{-1}(x + 3) + c \end{aligned}$$

### Exercise for Reader

Evaluate the following integrals.

1.  $\int \frac{dx}{\sqrt{2x-x^2}}$
2.  $\int \frac{dx}{x\sqrt{3x^2-2}}$
3.  $\int \frac{\cos x}{\sqrt{1-4\sin^2 x}} dx$
4.  $\int \frac{6e^x+5}{4e^x+7} dx$
5.  $\int \frac{3\sin x+2\cos x}{2\sin x+3\cos x} dx$
6.  $\int \frac{3x^2+2x+5}{x^2+4x+7} dx$