Lecture 15

Learning Objectives

At the end of this class, students should be able to:

- derive and use standard integrals
- solve related problems

Some more Standard Integrals

a) $\int \frac{1}{a^2+x^2} dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + c$

Let $x = at$ then $dx = a\, dt$

Thus, $\int \frac{1}{a^2+x^2} dx \quad = \int \frac{1}{a^2+a^2t^2} \times a\, dt$

$$= \frac{1}{a}\int \frac{1}{1+t^2} dt$$

$$= \frac{1}{a}\tan^{-1} t + c$$

$$= \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + c \qquad [\because x = at]$$

b) $\int \frac{1}{x^2-a^2} dx = \frac{1}{2a}\ln\left(\frac{x-a}{x+a}\right) + c,\ x > a$

Since $\frac{1}{x^2-a^2} = \frac{1}{(x-a)(x+a)} = \frac{1}{2a}\left[\frac{1}{(x-a)} - \frac{1}{(x+a)}\right]$

Thus, $\int \frac{1}{x^2-a^2} dx \quad = \frac{1}{2a}\int \left[\frac{1}{(x-a)} - \frac{1}{(x+a)}\right] dx$

$$= \frac{1}{2a}[\ln(x-a) - \ln(x+a)] + c$$

$$= \frac{1}{2a}\ln\left(\frac{x-a}{x+a}\right) + c$$

$\therefore \quad \int \frac{1}{x^2-a^2} dx = \frac{1}{2a}\ln\left(\frac{x-a}{x+a}\right) + c,\ x > a$

or, $\quad \int \frac{1}{x^2-a^2} dx = \frac{1}{2a}\ln\left|\frac{x-a}{x+a}\right| + c$

c) $\int \frac{1}{a^2-x^2} dx = \frac{1}{2a}\ln\left(\frac{a+x}{a-x}\right) + c,\ x < a$

Here, $\int \frac{1}{a^2-x^2} dx \quad = -\int \frac{1}{x^2-a^2} dx$

$$= -\frac{1}{2a}\ln\left|\frac{x-a}{x+a}\right| + c \qquad \left[\because \int \frac{1}{x^2-a^2} dx = \frac{1}{2a}\ln\left|\frac{x-a}{x+a}\right| + c\right]$$

$$= -\frac{1}{2a}\ln\left|\frac{-(a-x)}{a+x}\right| + c$$

$$= -\frac{1}{2a}\ln\left|\frac{a-x}{a+x}\right| + c$$

$$= \frac{1}{2a}\ln\left|\frac{a+x}{a-x}\right| + c \qquad \left[\because -\ln\left(\frac{p}{q}\right) = \ln\left(\frac{q}{p}\right)\right]$$

Integrals of the form $\int \frac{dx}{ax^2+bx+c}$

To express this integral in the standard form, we convert the quadratic expression $x^2 + bx + c$ in the form of sum or difference of two squares and then we can apply one of the above three formulas (a), (b), and (c). Let us try to understand from the following illustrations.

Illustration

Evaluate: $\int \frac{dx}{x^2+6x+8}$

Solution

Here, $x^2 + 6x + 8 = x^2 + 2 \times 3 \times x + 9 - 1 = (x+3)^2 - 1^2$

Thus, $\int \frac{dx}{x^2+6x+8} = \int \frac{dx}{(x+3)^2-1^2}$

$$= \frac{1}{2\times 1}\ln\left[\frac{(x+3)-1}{(x+3)+1}\right] + c$$

$$= \frac{1}{2}\ln\left[\frac{x+2}{x+4}\right] + c$$

Illustration

Evaluate: $\int \frac{dx}{15+4x-x^2}$

Solution

Here, $15 + 4x - x^2 = -(x^2 - 4x - 15)$

$$= -(x^2 - 2 \times 2 \times x + 4 - 19)$$

$$= -\left[(x-2)^2 - \left(\sqrt{19}\right)^2\right] = \left(\sqrt{19}\right)^2 - (x-2)^2$$

Thus, $\int \frac{dx}{15+4x-x^2} = \int \frac{dx}{\left(\sqrt{19}\right)^2-(x-2)^2}$

$$= \frac{1}{2\times\sqrt{19}}\ln\left[\frac{\sqrt{19}+(x-2)}{\sqrt{19}-(x-2)}\right] + c$$

$$= \frac{1}{2\sqrt{19}}\ln\left[\frac{\sqrt{19}-2+x}{\sqrt{19}+2-x}\right] + c$$

Illustration

Evaluate: $\int \frac{dx}{3x^2+2x+7}$

Solution

Here, $3x^2 + 2x + 7 = 3\left(x^2 + \frac{2}{3}x + \frac{7}{3}\right) \ = 3\left\{x^2 + 2 \times \frac{1}{3} \times x + \frac{1}{9} - \frac{1}{9} + \frac{7}{3}\right\}$

$$= 3\left\{\left(x + \frac{1}{3}\right)^2 + \frac{20}{9}\right\} \ = 3\left\{\left(x + \frac{1}{3}\right)^2 + \left(\frac{2\sqrt{5}}{3}\right)^2\right\}$$

Thus, $\int \frac{dx}{3x^2+2x+7} = \frac{1}{3}\int \frac{dx}{\left\{\left(x+\frac{1}{3}\right)^2+\left(\frac{2\sqrt{5}}{3}\right)^2\right\}}$

$$= \frac{1}{3} \times \frac{3}{2\sqrt{5}} \tan^{-1}\left\{\frac{x+1/3}{2\sqrt{5}/3}\right\} + c$$

$$= \frac{1}{2\sqrt{5}} \tan^{-1}\left\{\frac{3x+1}{2\sqrt{5}}\right\} + c$$

Illustration

Evaluate: $\int \frac{\sec^2 x}{25-16\ \tan^2 x} dx$

Solution

Let $4\tan x = t$ then $4\sec^2 x\, dx = dt \ \Rightarrow \sec^2 x\, dx = dt/4$

Thus, $\int \frac{\sec^2 x}{25-16\ \tan^2 x} dx \qquad = \frac{1}{4}\int \frac{dt}{5^2-t^2}$

$$= \frac{1}{4} \times \frac{1}{2\times 5} \ln\left(\frac{5+t}{5-t}\right) + c$$

$$= \frac{1}{40} \ln\left(\frac{5+4\tan x}{5-4\tan x}\right) + c$$

Integrals of the form $\int \frac{dx}{a+b\sin x+c\cos x}$, $\int \frac{dx}{a+b\sin x}$, $\int \frac{dx}{a+c\cos x}$

To express these integrals in the standard form, we make the use of following identities:

$\sin x = 2\sin\frac{x}{2}\cos\frac{x}{2}$

$\cos x = \cos^2\frac{x}{2} - \sin^2\frac{x}{2}$, and for the constant $'a'$, we write

$a = a\left(\sin^2\frac{x}{2} + \cos^2\frac{x}{2}\right)$

Let us try to understand from the following illustration.

Illustration

Evaluate: $\int \frac{dx}{3\sin x+2\cos x+3}$

Solution

Here, $\int \frac{dx}{3\sin x+2\cos x+3} \qquad = \int \frac{dx}{3\times 2\sin\frac{x}{2}\cos\frac{x}{2}+2\left(\cos^2\frac{x}{2}-\sin^2\frac{x}{2}\right)+3(\sin^2\frac{x}{2}+\cos^2\frac{x}{2})}$

$$= \int \frac{dx}{(\sin^2\frac{x}{2} + \mathbf{6}\sin\frac{x}{2}\cos\frac{x}{2} + \mathbf{5}\cos^2\frac{x}{2})}$$

Dividing numerator and denominator by $\cos^2\frac{x}{2}$, we get

$$= \int \frac{\sec^2\frac{x}{2}}{(\tan^2\frac{x}{2} + \mathbf{6}\tan\frac{x}{2} + 5)}\,dx$$

Substituting $\tan\frac{x}{2} = u$ than

$$\frac{1}{2}\sec^2\frac{x}{2}\,dx = du \quad \Rightarrow \sec^2\frac{x}{2}\,dx = 2du$$

Thus, $\int \frac{\sec^2\frac{x}{2}}{(\tan^2\frac{x}{2} + \mathbf{6}\tan\frac{x}{2} + 5)}\,dx \quad = 2\int \frac{1}{(u^2 + 6u + 5)}\,du$

$$= 2\int \frac{1}{(u^2 + 2\times3\times u + 9 - 4)}\,du$$

$$= 2\int \frac{1}{(u+3)^2 - 2^2}\,du$$

$$= \frac{2}{2\times2}\ln\left[\frac{(u+3)-2}{(u+3)+2}\right] + c$$

$$= \frac{1}{2}\ln\left[\frac{\tan\frac{x}{2} + 1}{\tan\frac{x}{2} + 5}\right] + c$$

Integral of the form $\int \frac{dx}{a\sin x + b\cos x}$

To express this integral in the standard form, we make the use of following substitutions:

$a = r\cos t$ and $b = r\sin t$ then $r^2 = a^2 + b^2 \Rightarrow r = \sqrt{a^2 + b^2}$ and $t = \tan^{-1}(b/a)$

$\therefore \quad a\sin x + b\cos x \quad = r\sin x\cos t + r\cos x\sin t$

$$= r\sin(x + t)$$

Illustration

Evaluate: $\int \frac{dx}{3\sin x + 2\cos x}$

Solution

Let $3 = r\cos t$ and $2 = r\sin t$ then $r = \sqrt{2^2 + 3^2} = \sqrt{13}$ and $t = \tan^{-1}(2/3)$

$\therefore \quad 3\sin x + 2\cos x \quad = r\sin x\cos t + r\cos x\sin t$

$$= r\sin(x + t)$$

Thus, $\int \frac{dx}{3\sin x + 2\cos x} \quad = \frac{1}{r}\int \frac{dx}{\sin(x+t)}$

$$= \frac{1}{r}\int \operatorname{cosec}(x + t)\,dx$$

$$= \frac{1}{r}\ln\left[\tan\left(\frac{x+t}{2}\right)\right] + c$$

$$= \frac{1}{\sqrt{13}} \ln \left[\tan\left(\frac{x}{2} + \frac{1}{2}\tan^{-1}\frac{2}{3}\right)\right] + c$$

Exercise for Reader

Evaluate the following integrals.

1. $\int \frac{dx}{4-9x^2}$
2. $\int \frac{dx}{4x^2+25}$
3. $\int \frac{dx}{x^2+6x+7}$
4. $\int \frac{dx}{3+2x-x^2}$
5. $\int \frac{dx}{4-3x-2x^2}$
6. $\int \frac{dx}{3x^2+2x+7}$
7. $\int \frac{\cos x}{9\sin^2 x+12\sin x+5} dx$
8. $\int \frac{dx}{2-3\sin x}$
9. $\int \frac{dx}{1+\sin x+\cos x}$
10. $\int \frac{dx}{4\sin x+3\cos x}$
11. $\int \frac{dx}{4+5\cos x}$