**Lecture 14**

Learning Objectives
At the end of this class, students should be able to:

- use various techniques of integration
- solve related problems

## Techniques of Integration

If the function to be integrated is in the standard form, we can easily integrate it by using the formulas mentioned in previous lecture. When the integrand is not in the standard form, we have to convert it in the standard form. The following techniques of integration are available to us.

1. Method of simplification
2. Method of substitution
3. Method of integration by parts
4. Method of partial fraction

## 1. Method of Simplification

If the integrand is the product or the quotient of two functions. In this case we try to express it as a sum of two or more terms and integrate. Let us try to understand with the help of the following illustrations.

*Illustration*

Evaluate: $\int \frac{x+3}{x-1} dx$

*Solution*

The integrand is the quotient of two functions $x + 3$ and $x - 1$. The integral can be written as

$$\int \frac{x+3}{x-1} dx = \int \frac{x+4-1}{x-1} dx$$

[While rewriting the numerator, we need to take care of the denominator]

$$= \int \left[\frac{x-1}{x-1} + \frac{4}{x-1}\right] dx$$

$$= \int \left[1 + \frac{4}{x-1}\right] dx$$

$$= \int dx + 4 \int \frac{1}{x-1} dx$$

$$= x + 4 \ln(x - 1) + c$$

***Note:*** Whenever numerator is of equal or higher degree than the denominator. We divide the numerator by denominator until a remainder is obtained which is of lower degree than the denominator,

*i.e.,* $\frac{p(x)}{q(x)} = Quotient + \frac{Remainder}{q(x)}$ ,

where $p(x)$ and $q(x)$ are polynomials and the degree of $p$ is greater than or equal to that of $q$.

*Illustration*

Evaluate: $\int \frac{x^2+3x+5}{x+1} dx$.

*Solution*

*Here,* $\frac{x^2+3x+5}{x+1} = (x+2) + \frac{3}{x+1}$ *, then*

$$\begin{aligned}\int \frac{x^2+3x+5}{x+1} dx &= \int \left[(x+2) + \frac{3}{x+1}\right] dx \\ &= \int (x+2)dx + 3\int \frac{1}{x+1} dx \\ &= \left(\frac{x^2}{2} + 2x\right) + 3\ln(x+1) + c\end{aligned}$$

*If there is surd in the denominator, we first rationalize it and then integrate. Let us try to understand with the help of the following illustration.*

*Illustration*

Evaluate: $\int \frac{1}{\sqrt{x+1}-\sqrt{x}} dx$

*Solution*

The integrand contains a surd in the denominator; therefore, first we rationalize the expression in the denominator.

$$\begin{aligned}\int \frac{1}{\sqrt{x+1}-\sqrt{x}} dx &= \int \frac{1}{\sqrt{x+1}-\sqrt{x}} \times \frac{\sqrt{x+1}+\sqrt{x}}{\sqrt{x+1}+\sqrt{x}} dx \\ &= \int \frac{\sqrt{x+1}+\sqrt{x}}{x+1-x} dx \\ &= \int (\sqrt{x+1} + \sqrt{x}) dx \\ &= \frac{(x+1)^{3/2}}{3/2} + \frac{(x)^{3/2}}{3/2} + c \\ &= \frac{2}{3}\left((x+1)^{3/2} + x^{3/2}\right) + c\end{aligned}$$

**2. Method of Substitution**

Sometimes, the integrand is not in the appropriate form so that we cannot use the integration formulas directly. In this situation, we need to change the variable by suitable substitution.
We follow the following procedure:

Step 1: Select a substitution that appears to simplify the integrand. In particular, try to select *u* so that *du* is a factor in the integrand.
Step 2: Express the integrand entirely in terms of *u* and *du*, completely eliminating the original variable.
Step 3: Evaluate the new integral if possible.
Step 4: Express the answer found in step 3 in terms of the original variable.

## i) When the integrand is of the form $f'(ax+b)$

In this case, we put $ax + b = u$ and $dx = \frac{du}{a}$

Thus $\int f'(ax+b)\,dx = \frac{1}{a}\int f'(u)du$

$$= \frac{f(u)}{a} + c$$

$$= \frac{f(ax+b)}{a} + c$$

Let us try to understand with the help of the following illustrations.

*Illustration*

Evaluate: $\int \frac{dx}{(2-5x)^3}$.

*Solution*

Let $2 - 5x = u$. Then

$-5dx = du$

or $dx = -du/5$

$$\therefore \int \frac{dx}{(2-5x)^3} = \int \frac{(-du/5)}{u^3}$$

$$= -\frac{1}{5}\int u^{-3}du$$

$$= -\frac{1}{5}\frac{u^{-2}}{-2} + c$$

$$= \frac{1}{10u^2} + c$$

$$= \frac{1}{10(2-5x)^2} + c$$

In general, when the expression within bracket (here, $ax + b$) is linear in $x$.

1. $\int (ax+b)^n dx = \frac{(ax+b)^{n+1}}{a(n+1)} + c,$ for $n \neq -1$, and
2. $\int \frac{1}{(ax+b)}dx = \frac{1}{a}ln|(ax+b)| + c$

We can use these results as formulas.

*Illustration*

Evaluate: $\int (7x+5)^{4/9}dx$.

*Solution*

Here, $\int (7x+5)^{4/9}dx = \frac{(7x+5)^{\frac{4}{9}+1}}{7\times\left(\frac{4}{9}+1\right)} + c$

$$= \frac{9}{91}(7x+5)^{\frac{13}{9}} + c$$

**ii) When the integrand is of the form $[f(x)]^n f'(x)$**

Here $[f(x)]^n$ is the power of $f(x)$ and $f'(x)$ is the derivative of $f(x)$. In such a case, we put $f(x) = u$ so that $f'(x)\,dx = du$.

Thus, $\int [f(x)]^n f'(x)dx \quad = \int u^n du$

$$= \frac{u^{n+1}}{n+1} + c, \qquad n \neq -1$$

$$= \frac{[f(x)]^{n+1}}{n+1} + c$$

Let us try to understand with the help of the following illustrations.

*Illustration*

Evaluate: $\int 14x\sqrt{2 + 7x^2}dx$

*Solution*

Let $2 + 7x^2 = u$. Then

$14x\,dx = du$

or, $dx = du/(14x)$

$\therefore \quad \int 14x\sqrt{2 + 7x^2}\,dx \quad = \int 14x\sqrt{u}\,du/(14x)$

$$= \int u^{1/2}\,du$$

$$= \left(\frac{u^{3/2}}{3/2}\right) + \text{c}$$

$$= \frac{2}{3}u^{3/2} + \text{c}$$

$$= \frac{2}{3}(2 + 7x^2)^{3/2} + \text{c}$$

**iii) When the integrand is of the form $\frac{f\prime(x)}{f(x)}$**

Here the numerator is the derivative of the denominator. In this case, we put $f(x) = u$ so that $f'(x)dx = du$.

Thus, $\int \frac{f\prime(x)}{f(x)}dx = \int \frac{1}{u}du = \ln u + c = \ln f(x) + c$

Let us try to understand with the help of the following illustrations.

*Illustration*

Evaluate: $\int \frac{e^{2x}+e^{-2x}}{e^{2x}-e^{-2x}}dx$

*Solution*

Let $e^{2x} - e^{-2x} = u$. Then $2(e^{2x} + e^{-2x})\,dx = du \Rightarrow (e^{2x} + e^{-2x})\,dx = du/2$

$\therefore \quad \int \frac{e^{2x}+e^{-2x}}{e^{2x}-e^{-2x}}dx \quad = \int \frac{1}{u}\frac{du}{2}$

$$= \frac{1}{2} \ln u + c = \frac{1}{2} \ln( e^{2x} - e^{-2x}) + c$$

*Illustration*

Evaluate: $\int \frac{e^x - 1}{e^x + 1} dx$

*Solution*

Here, $\int \frac{e^x - 1}{e^x + 1} dx = \int \frac{e^{x/2} - e^{-x/2}}{e^{x/2} + e^{-x/2}} dx$

Now, it can be easily solved by substitution technique.

*Illustration*

Evaluate: $\int \tan x \, dx$

*Solution*

We have $\int \tan x \, dx \quad = \int \frac{\sin x}{\cos x} dx$

Let $\cos x = u$. Then $-\sin x \, dx = du \Rightarrow \sin x \, dx = -du$

$$\therefore \int \frac{\sin x}{\cos x} dx = \int \frac{1}{u} (-du)$$

$$= -\ln u + c$$

$$= -\ln( \cos x) + c$$

$$= \ln( \cos x)^{-1} + c$$

$$= \ln( \sec x) + c$$

*Illustration*

Evaluate: $\int \frac{1}{e^x + 1} dx$

*Solution*

Here $\int \frac{1}{e^x + 1} dx = \int \frac{1}{e^x(1 + e^{-x})} dx = \int \frac{e^{-x}}{(1 + e^{-x})} dx$

Let $1 + e^{-x} = u$. Then $-e^{-x} dx = du$

or $e^{-x} dx = -du$

$$\therefore \int \frac{e^{-x}}{(1 + e^{-x})} dx = \int \frac{1}{u} (-du)$$

$$= -\ln u + c$$

$$= -\ln( 1 + e^{-x}) + c$$

**iv) When the integrand is of the form $f(x)[g(x)]^n$ or $\frac{f(x)}{[g(x)]^n}$**

Sometimes we can integrate functions of the above forms by putting $f(x) = u$ even though neither function is the derivative of the other. Let us try to understand with the help of the following illustrations.

*Illustration*

Evaluate: $\int 5x\sqrt{ax+b}dx$

*Solution*

Let $ax + b = u$. Then $a\,dx = du$ and $x = \frac{u-b}{a}$

$$\therefore \int 5x\sqrt{ax+b}dx = \int 5\left(\frac{u-b}{a}\right)\sqrt{u}\frac{du}{a}$$

$$= \frac{5}{a^2}\int (u^{3/2} - bu^{1/2})du$$

$$= \frac{5}{a^2}\left[\frac{2}{5}u^{5/2} - \frac{2}{3}bu^{3/2}\right] + c$$

$$= \frac{10}{a^2}\left[\frac{1}{5}(ax+b)^{5/2} - \frac{1}{3}b(ax+b)^{3/2}\right] + c$$

*Illustration*

Evaluate: $\int \frac{x\,dx}{\sqrt{x+3}}$

*Solution*

Let $x + 3 = u^2$. Then

$dx = 2u\,du$ and $x = u^2 - 3$

$$\therefore \int \frac{x\,dx}{\sqrt{x+3}} = \int \frac{(u^2-3)2u\,du}{u}$$

$$= 2\int (u^2-3)du = 2(\frac{u^3}{3} - 3u) + c$$

$$= \frac{2}{3}(x+3)^{3/2} - 6(x+3)^{1/2} + c$$

**v) When the integrand involves exponential or logarithmic functions**

Let us try to understand with the help of the following illustrations.

*Illustration*

Evaluate: $\int x^2 e^{x^3-5}dx$

*Solution*

Let $x^3 - 5 = u$. Then $3x^2dx = du$

or $dx = \frac{du}{3x^2}$

$$\therefore \int x^2 e^{x^3-5}dx = \int x^2 e^u \frac{du}{3x^2}$$

$$= \frac{1}{3}\int e^u du$$

$$= \frac{1}{3} e^{u} + c$$
$$= \frac{1}{3} e^{x^3 - 5} + c$$

*Illustration*

Evaluate: $\int \frac{(a+b \ln x)^4}{x} dx$

*Solution*

Let $a + b\, ln\, x = u$. Then $\frac{b}{x} dx = du$

or $dx = \frac{x}{b} du$

$\therefore \int \frac{(a+b \ln x)^4}{x} dx = \int \frac{u^4}{x} \times \frac{x}{b} du$

$$= \frac{1}{b} \int u^4 du$$
$$= \frac{1}{5b} u^5 + c$$
$$= \frac{1}{5b} (a + b \ln x)^5 + c$$

**Exercise for Reader**

Evaluate the following integrals.

1. $\int \frac{x^2+5x+7}{x+2} dx$
2. $\int \frac{1}{\sqrt{2x+1} - \sqrt{2x-3}} dx$
3. $\int (5x + 2)^{-3} dx$
4. $\int 2x(x^2 + 5)^6 dx$
5. $\int \frac{\cos x}{\sqrt{1+\sin x}} dx$
6. $\int \cot^3 x \operatorname{cosec}^3 x\, dx$
7. $\int x^2 e^{x^3 - 1} dx$
8. $\int \frac{(ln\, x)^2}{3x} dx$
9. $\int \frac{dx}{\sqrt{e^{2x} - 1}}$
10. $\int \frac{e^{m \tan^{-1} x}}{1+x^2}\ dx$