Lecture 13

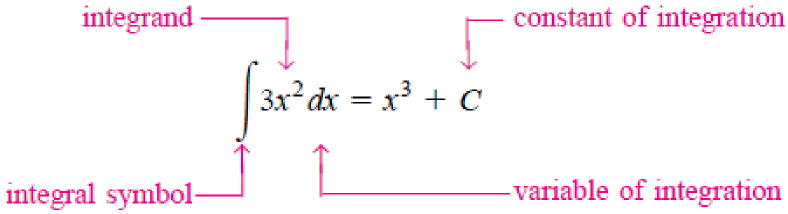
Learning Objectives
At the end of this class, students should be able to:

- understand the concept of indefinite integral
- use integration rules
- solve related problems

Integration

The reverse process of differentiation is known as integration. We use the symbol $\int f(x)\,dx$ to represent the integration of $f(x)$. We write $\int f(x)\,dx = F(x) + c$ if $F'(x) = f(x)$. This integral is called indefinite integral because it involves a constant c that can take on any value. The symbol $\int$ is called an integral sign and the function $f(x)$ is called the integrand. The symbol dx indicates that the integration is performed with respect to the variable x. The letter c is called the constant of integration.

These features are displayed in the following diagram for the indefinite integral of $f(x) = 3x^2$:



Note: While finding integral, if $F'(x) = f(x)$, then the integration $\int f(x)dx = F(x) + c$ is correct, but if $F'(x)$ is anything other than $f(x)$, we must have committed a mistake.

Rules for Integrating Common Functions

1. The constant rule: $\int k\,dx = kx + c$ for constant k
2. The power rule: $\int x^n\,dx = \frac{x^{n+1}}{n+1} + c$ for all $n \neq -1$
3. The logarithmic rule: $\int \frac{1}{x} dx = \ln|x| + c$ for all $x \neq 0$
4. The exponential rule: $\int e^{kx}\,dx = \frac{e^{kx}}{k} + c$ for constant $k \neq 0$
5. $\int \sin mx\,dx = -\frac{1}{m}\cos mx + c$
6. $\int \cos mx\,dx = \frac{1}{m}\sin mx + c$
7. $\int \sec^2 mx\,dx = \frac{1}{m}\tan mx + c$
8. $\int \operatorname{cosec}^2 mx\,dx = -\frac{1}{m}\cot mx + c$

9. $\int \sec mx \tan mx \, dx = \frac{1}{m} \sec mx + c$

10. $\int \operatorname{cosec} mx \cot mx \, dx = -\frac{1}{m} \operatorname{cosec} mx + c$

11. $\int \frac{1}{\sqrt{1-x^2}} dx = \sin^{-1} x + c$

12. $\int \frac{1}{1+x^2} dx = \tan^{-1} x + c$

13. $\int \frac{1}{x\sqrt{x^2-1}} dx = \sec^{-1} x + c$

14. $\int \sinh x \, dx = \cosh x + c$

15. $\int \cosh x \, dx = \sinh x + c$

16. $\int \operatorname{sech}^2 x \, dx = \tanh x + c$

17. $\int \operatorname{cosech}^2 x \, dx = -\coth x + c$

18. $\int \operatorname{cosech} x \coth x \, dx = -\operatorname{cosech} x + c$

19. $\int \operatorname{sech} x \tanh x \, dx = -\operatorname{sech} x + c$

Illustration

Evaluate the following integrals:

a) $\int 5dx$

b) $\int 3x^4 dx$

Solution

a) $\int 5dx = 5x + c$ Check: $\frac{d}{dx}(5x + c) = 5$

b) $\int 3x^4 dx = \frac{3}{5}x^5 + c$ Check: $\frac{d}{dx}(\frac{3}{5}x^5 + c) = 3x^4$

Properties of Integration

1. A constant factor can be moved to the front of an indefinite integral:

$$\int [k \, . \, f(x)] dx = k \int f(x) dx$$

2. The integral of a sum or a difference is the sum or the difference of the integral:

$$\int [f(x) \pm g(x)] \, dx = \int f(x) dx \pm \int g(x) dx$$

Illustration

Evaluate the following integrals:

a) $\int (x^{1/3} - 3x^{-2/3} + 6) dx$

b) $\int \left(\frac{1}{2x} - \frac{2}{x^2} + \frac{3}{\sqrt{x}} \right) dx$

c) $\int \left(3e^u + \frac{6}{u} + \ln 2\right) du$

Solution

a) $\int (x^{1/3} - 3x^{-2/3} + 6)dx$

$$= \int x^{1/3}dx - 3\int x^{-2/3}dx + 6\int dx$$
$$= \frac{3}{4}x^{4/3} - 9x^{1/3} + 6x + c$$

b) $\int \left(\frac{1}{2x} - \frac{2}{x^2} + \frac{3}{\sqrt{x}}\right) dx$

$$= \frac{1}{2}\int \frac{1}{x}dx - 2\int x^{-2}dx + 3\int x^{-1/2}dx$$
$$= \frac{1}{2}\,ln\, x - \frac{2}{(-1)}x^{-1} + 6x^{1/2} + c$$
$$= \frac{1}{2}\,ln\, x + \frac{2}{x} + 6x^{1/2} + c$$

c) $\int \left(3e^u + \frac{6}{u} + \ln 2\right) du$

$$= 3\int e^u\, du + 6\int \frac{1}{u}du + \ln 2 \int du$$
$$= 3e^u + 6\ln u + u\ln 2 + c$$

Integration Using Trigonometric Identities

When the integrand involves trigonometric functions, it is sometimes possible to convert the integrand into a standard form, by applying algebraic operations and/or trigonometric identities.

Illustration

Evaluate $\int \frac{\sin x}{1+\sin x} dx$

Solution

Here, $\int \frac{\sin x}{1+\sin x} dx$

$$= \int \frac{\sin x}{1+\sin x} \times \frac{1-\sin x}{1-\sin x} dx$$
$$= \int \frac{\sin x - \sin^2 x}{1-\sin^2 x} dx$$
$$= \int \frac{\sin x - \sin^2 x}{\cos^2 x} dx$$
$$= \int \left[\frac{\sin x}{\cos^2 x} - \frac{\sin^2 x}{\cos^2 x}\right] dx$$
$$= \int [\sec x\,.\tan x - \tan^2 x]dx$$
$$= \int [\sec x\,.\tan x - \sec^2 x + 1]dx$$
$$= \int \sec x\,.\tan x\, dx - \int \sec^2 x\, dx + \int dx$$
$$= \sec x - \tan x + x + c$$

Illustration

Evaluate the following integrals:

a) $\int \frac{dx}{1-\cos 2x}$

b) $\int \sin^2 x \cos^2 x\, dx$

Solution

a) $\int \frac{dx}{1-\cos 2x} = \int \frac{dx}{1-(1-2sin^2 x)}$

$= \int \frac{dx}{2sin^2 x}$

$= \frac{1}{2}\int cosec^2\, x\, dx$

$= -\frac{1}{2}\cot x + c$

b) $\int \sin^2 x \cos^2 x\, dx = \frac{1}{4}\int (\sin 2x)^2 dx$

$= \frac{1}{4}\int \sin^2 2x\, dx$

$= \frac{1}{4}\int \frac{1}{2}(1-\cos 4x)dx \qquad \left[\because \sin^2 x = \frac{1}{2}(1-\cos 2x)\right]$

$= \frac{1}{8}\left(x - \frac{1}{4}\sin 4x\right) + c$

The following table provides few examples of trigonometric functions that they can be converted into standard form(s) by using simple algebraic operations and trigonometric identities.

S. No.	Given Trigonometric Function(s)	Operations Involved in Converting the Function(s) to the Standard Form
1.	$\frac{\sin x}{\cos^2 x}$	$= \frac{1}{\cos x}\cdot\frac{\sin x}{\cos x} = \sec x \cdot \tan x$
2.	$\frac{\cos x}{\sin^2 x}$	$= \frac{1}{\sin x}\cdot\frac{\cos x}{\sin x} = \text{cosec}\, x \cdot \cot x$
3.	$\frac{1}{1+\sin x}$	$= \frac{1}{1+\sin x}\cdot\frac{1-\sin x}{1-\sin x} = \frac{1-\sin x}{\cos^2 x}$ $= \sec^2 x - \sec x \cdot \tan x$
4.	$\frac{1}{1-\sin x}$	$= \frac{1+\sin x}{\cos^2 x} = \sec^2 x + \sec x \cdot \tan x$
5.	$\frac{1}{1+\cos x}$	$= \frac{1}{1+\cos x}\cdot\frac{1-\cos x}{1-\cos x} = \frac{1-\cos x}{\sin^2 x}$ $= \text{cosec}^2 x - \text{cosec}\, x \cdot \cot x$
6.	$\frac{1}{1-\cos x}$	$= \frac{1+\cos x}{\sin^2 x} = \text{cosec}^2 x + \text{cosec}\, x \cdot \cot x$

Illustration

Evaluate: $\int \sin 3x \cos 4x\, dx$

Solution

Here, $\int \sin 3x \cos 4x \, dx$ $= \frac{1}{2}\int 2\sin 3x \cos 4x \, dx$

$$= \frac{1}{2}\int [\sin(3x + 4x) + \sin(3x - 4x)]dx$$

$$[\because 2 \sin A \cos B = \sin(A + B) + \sin(A - B)]$$

$$= \frac{1}{2}\int (\sin 7x - \sin x)dx$$

$$= \frac{1}{2}\int \sin 7x \, dx - \frac{1}{2}\int \sin x \, dx$$

$$= \frac{1}{2}\left(\frac{-\cos 7x}{7} + \cos x\right) + c$$

Exercise for Reader

Evaluate the following integrals.

a) $\int 23dx$

b) $\int x^{1/2} \, dx$

c) $\int x^{-4/5}dx$

d) $\int (2 - 4x + 7x^2 - x^3)dx$

e) $\int \frac{ax^2+bx+c}{x} dx$

f) $\int \left(\sqrt{x} + \frac{1}{\sqrt{x}}\right)^2 dx$

g) $\int \frac{e^{2x}+e^x+1}{e^x} dx$

h) $\int \frac{\cos x}{1+\cos x} dx$

i) $\int \sin 5x \cos 3x \; dx$

j) $\int \frac{\cos 2x}{\sin^2 x \cos^2 x} \; dx$