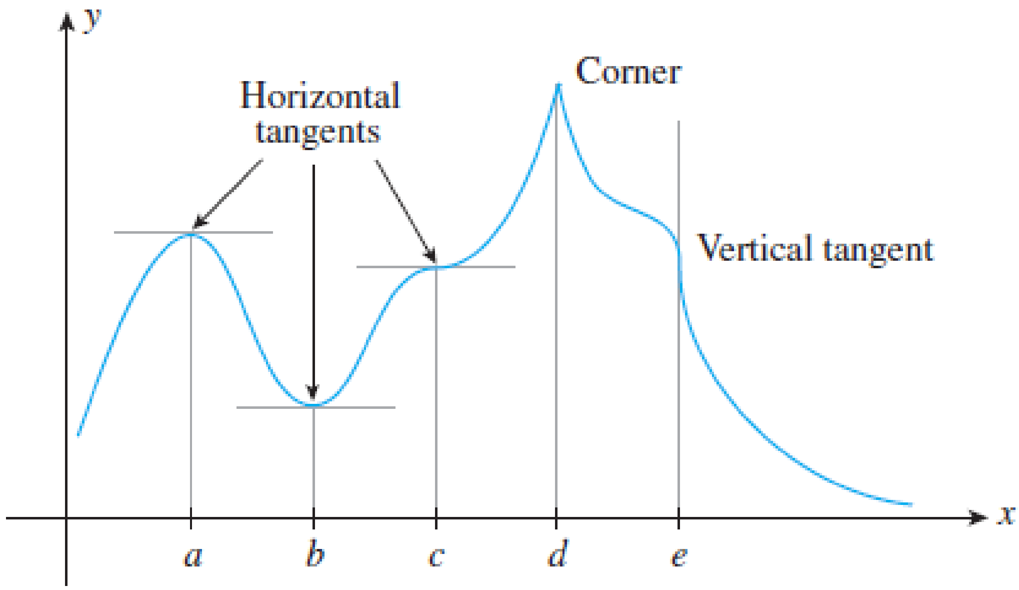
Lecture 12

Learning Objectives
At the end of this class, students should be able to:

- understand the concept critical point
- familiar with second derivative test to identify extreme values of the function
- solve related problems

Critical Values

A critical value of a function f is any number x in the domain of f such that $f'(x) = 0$ or $f'(x)$ does not exist. Let us consider the following graph of a function $f(x)$.



In the above figure, the graph of a function has critical values at a, b, c, d, and e. We observe that $f'(x) = 0$at a, b, and c. $f'(x)$ is not defined at d because the function has a corner at that point. Similarly, $f'(x)$ does not exist at e because the tangent line is vertical at that point.

A 'peak' on the graph of a function f is known as a relative maximum of f, and a 'valley' is a relative minimum. Thus, a relative maximum is a point on the graph of f that is at least as high as any nearby point on the graph, while a relative minimum is at least as low as any nearby point.

In the above figure, relative maximum occurs at a and d whereas relative minimum occurs at b. We have to apply standard test to identify whether the function has relative maximum or minimum at the particular critical point.

The Second Derivative Test

The following steps will be followed for identifying optimum values of the function.

1. Find all critical values x^*, such that $f'(x^*) = 0$.
2. For any critical value x^*, determine the value of $f''(x^*)$.

a) If $f''(x^*) > 0$, the given function has a relative minimum at x^*.
b) If $f''(x^*) < 0$, the given function has a relative maximum at x^*.
c) If $f''(x^*) = 0$, no conclusion can be drawn. Another test is required.

Illustration

Examine the function $f(x) = x^3 - 6x^2 + 9x + 5$ for any critical points and determine their nature.

Solution

We have $f(x) = x^3 - 6x^2 + 9x + 5$ then

$$f'(x) = 3x^2 - 12x + 9$$

For critical points, setting $f'(x) = 0$

i.e. $3x^2 - 12x + 9 = 0$

or, $3(x-1)(x-3) = 0$

Thus, the critical points occur at $x = 1$ and $x = 3$.

Now, $f''(x) = 6x - 12$

When $x = 1$,

$$f''(1) = 6 \times 1 - 12 = -6 < 0$$

Thus, the given function has a relative maximum at $x = 1$.

When $x = 3$,

$$f''(3) = 6 \times 3 - 12 = 6 > 0$$

Thus, the given function has a relative minimum at $x = 3$.

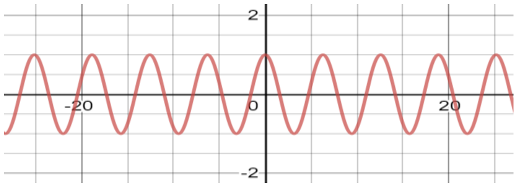
Illustration

Examine the function $f(x) = \cos x$ for any critical points and determine their nature.

Solution

We have $f(x) = \cos x$ then

$$f'(x) = -\sin x$$



For critical points, setting $f'(x) = 0$

i.e. $-\sin x = 0$

or, $\sin x = 0 \quad \Rightarrow x = n\pi$

Thus, the critical points occur at $x = n\pi$

Now, $f''(x) = -\cos x$

When $x = n\pi$, $f''(x) = -\cos n\pi$

When n is even, i.e., $n = 0, \pm 2, \pm 4, \ . \ . \ .$

$$f''(x) = -\text{ve}$$

Thus, the given function has maximum value.

When n is odd, i.e., $n = \pm 1, \pm 3, \pm 5, \ . \ . \ .$

$$f''(x) = +\text{ve}$$

Thus, the given function has minimum value.

Illustration

A rectangular garden of area 75 square feet is to be surrounded on three sides by a brick wall costing \$10 per foot and on one side by a fence costing \$5 per foot. Find the dimensions of the garden such that the cost of material is minimized.

Solution

Let x feet and y feet be the two sides of the rectangular garden.

Then, according to the question, the area of the garden $xy = 75$ (i)

Now, the fencing cost of three sides by brick wall is given by $10(2x + y)$.

Similarly, fencing cost of the remaining one side is given by $5y$.

Therefore, the total fencing cost will be $10(2x + y) + 5y = 20x + 15y = T$ (say)

Now, we have to find the values of x and y for minimum T.

Using equation (i), we can write $T = \frac{1500}{y} + 15y$

Then $T' = -\frac{1500}{y^2} + 15$

For minimum, $T' = 0$

i.e., $-\frac{1500}{y^2} + 15 = 0$

or, $y^2 = 100$

or, $y = \pm 10$

The value of y cannot be negative, so we neglect $y = -10$.

Now, $T'' = \frac{3000}{y^2}$

When $y = 10$, $T'' = \frac{3000}{10^2} > 0$

Thus, the total fencing cost T is minimum for $y = 10$.

From equation (1), $x = 7.5$ and the minimum cost

$$T = \frac{1500}{10} + 15 \times 10 = 300$$

Thus, the required dimensions of the garden are 7.5 feet and 10 feet, and the minimum cost of the materials is \$300.

Illustration

Prove that the greatest rectangle that can be inscribed in the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ with area $2ab$.

Solution

We have the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. Its center exists at (0, 0). This equation can be written as

$$y^2 = \frac{a^2b^2 - b^2x^2}{a^2} \qquad \text{(i)}$$

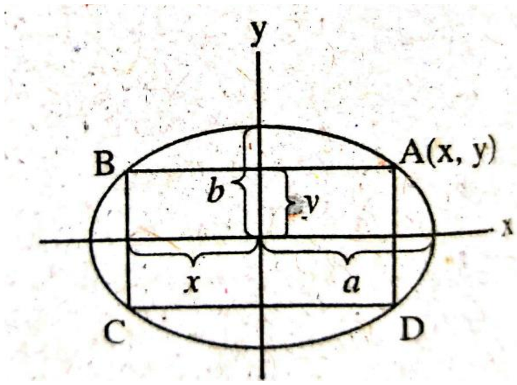
Let A, B, C, D be the vertices of the rectangle inscribed in the ellipse. Let A(x, y) be on the ellipse then the sides of rectangle are $2x$ and $2y$.

The area of rectangle is $A = 4xy$

Now, $A^2 = 16x^2y^2$. Using equation (i), we have

$$A^2 = 16x^2\left(\frac{a^2b^2 - b^2x^2}{a^2}\right)$$

or, $$A^2 = 16b^2x^2 - \frac{16b^2x^4}{a^2}$$



Differentiating with respect to x, we get

$$\frac{d(A^2)}{dx} = 32b^2x - \frac{64b^2x^3}{a^2}$$

For critical points, setting $\frac{d(A^2)}{dx} = 0$

i.e. $$32b^2x - \frac{64b^2x^3}{a^2} = 0$$

or, $$x = a/\sqrt{2}$$

Now, $$\frac{d^2(A^2)}{dx^2} = 32b^2 - \frac{192b^2x^2}{a^2}$$

when $x = a/\sqrt{2}$,

$$\frac{d^2(A^2)}{dx^2} = 32b^2 - \frac{192b^2}{a^2} \times \frac{a^2}{2} = -64b^2 < 0$$

Thus, A^2 has maximum value when $x = a/\sqrt{2}$. Hence the area is maximum when $x = a/\sqrt{2}$.

The maximum area is $A = 4xy = 4 \times \frac{a}{\sqrt{2}} \times \frac{a}{\sqrt{2}} = 2ab$

[when $x = a/\sqrt{2}$, $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, gives $y = a/\sqrt{2}$]

Illustration

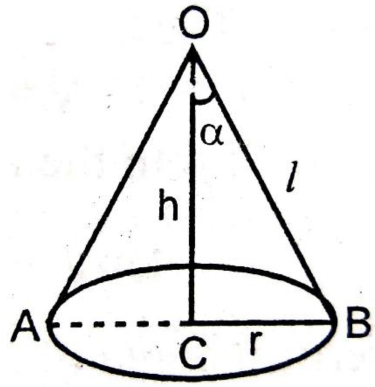
Show that the semi-vertical angle of a cone of maximum volume and of given slant height is $\tan^{-1}\sqrt{2}$.

Solution

Let OAB be a cone having height *h* and radius of the base *r*. Let *l* be the slant height and α be the semi-vertical angle. Then

$$\sin\alpha = \frac{r}{l} \text{ and } \cos\alpha = \frac{h}{l}$$

Thus, $r = l\sin\alpha$ and $h = l\cos\alpha$



Let V be the volume of a cone, then

$$V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi \times (l\sin\alpha)^2 \times (l\cos\alpha)$$

$$= \frac{1}{3}\pi l^3 \sin^2\alpha\cos\alpha$$

Now, $$\frac{dV}{d\alpha} = \frac{1}{3}\pi l^3[-\sin^2\alpha\sin\alpha + 2\sin\alpha\cos\alpha\cos\alpha]$$

$$= \frac{1}{3}\pi l^3[-\sin^3\alpha + 2\sin\alpha\cos^2\alpha]$$

For maximum or minimum, setting $\frac{dV}{d\alpha} = 0$

i.e. $$\frac{1}{3}\pi l^3[-\sin^3\alpha + 2\sin\alpha\cos^2\alpha] = 0$$

or, $\quad -\sin\alpha\,[\sin^2\alpha - 2\cos^2\alpha] = 0$

so, either $\sin\alpha = 0 \quad \Rightarrow \alpha = 0$ which is not possible.

or, $\quad \sin^2\alpha - 2\cos^2\alpha = 0 \qquad \Rightarrow \tan^2\alpha = 2$

or, $\quad \tan\alpha = \sqrt{2} \quad \Rightarrow \alpha = \tan^{-1}\sqrt{2}$

Again, $\frac{d^2V}{d\alpha^2} = \frac{1}{3}\pi l^3[-3\sin^2\alpha\cos\alpha - 4\sin^2\alpha\cos\alpha + 2\cos^3\alpha]$

$= \frac{1}{3}\pi l^3[-7\sin^2\alpha\cos\alpha + 2\cos^3\alpha]$

$= \frac{1}{3}\pi l^3\cos^3\alpha\,[-7\tan^2\alpha + 2]$

When $\tan\alpha = \sqrt{2} \quad \Rightarrow \cos\alpha = 1/\sqrt{3}$;

$\frac{d^2V}{d\alpha^2} = \frac{1}{3}\pi l^3 \times \frac{1}{3\sqrt{3}}[-7\times 2 + 2]$

$= -\frac{4\pi l^3}{3\sqrt{3}} < 0$

Thus, the volume is maximum when $\alpha = \tan^{-1}\sqrt{2}$

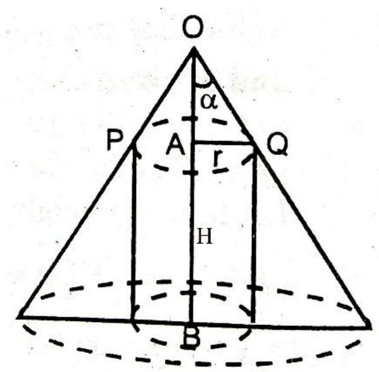
Illustration

Find the altitude of the right circular cylinder of maximum volume that can be inscribed in a given right circular cone of height h.

Solution

Let a cylinder of height H and radius r be inscribed in a given cone of height h. Let V be the volume of the cylinder. Then

$V = \pi r^2 H \qquad$ (i)



From the right-angled triangle ΔOAQ,

$\tan\alpha = \frac{\mathrm{AQ}}{\mathrm{OA}} = \frac{r}{\mathrm{OB}-\mathrm{AB}}$

or, $\quad \tan\alpha = \frac{r}{h-H} \quad \Rightarrow r = (h-H)\tan\alpha$

Substituting the value in equation (i), we get

$V = \pi(h-H)^2 H\tan^2\alpha$

Differentiating with respect to H, we get

$\frac{dV}{dH} = \pi(h-H)^2\tan^2\alpha + 2\pi(h-H)\times(-1)H\tan^2\alpha$

or, $\quad \frac{dV}{dH} = \pi\tan^2\alpha\,(h-H)(h-3H)$

For maximum or minimum $\frac{dV}{dH} = 0$

i.e. $\quad \pi\tan^2\alpha\,(h-H)(h-3H) = 0$

Since $\tan\alpha \neq 0,\ (h-H) \neq 0,\ \text{So } (h-3H) = 0$

$\Rightarrow H = h/3$

Again, $\frac{d^2V}{dH^2} = \pi\tan^2\alpha\,[(h-H)(-3) + (-1)(h-3H)]$

$= \pi\tan^2\alpha\,(-4h + 6H)$

When $H = h/3$,

$$\frac{d^2V}{dH^2} = \pi \tan^2 \alpha\,(-4h + 2h) = -2\pi h \tan^2 \alpha < 0$$

Thus, the volume is maximum when $H = h/3$.

Exercise for Reader

1. Examine the function $f(x) = \frac{\log x}{x}$ for any critical points and determine their nature.
2. Examine the function $f(x) = x^5 - 5x^4 + 5x^3 - 10$ for any critical points and determine their nature.
3. A cylindrical tin can close at both ends and of given capacity, has to be constructed. Show that the amount of tin required will be minimum when the height is equal to diameter.
4. The strength of a beam varies jointly as its breadth and the square of the depth. Find the dimensions of a strongest beam that can be cut from a circular wooden log.
5. Show that the maximum rectangle that can be inscribed in a circle is a square.