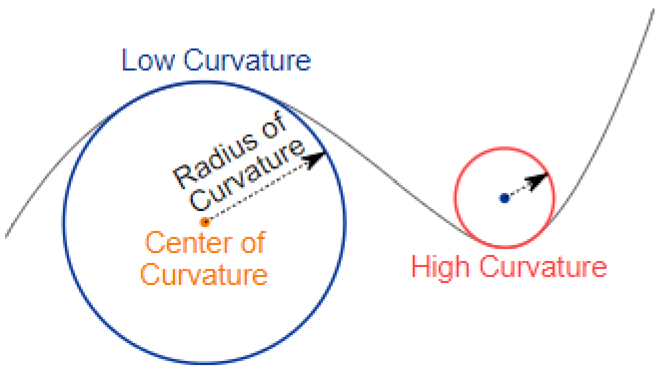
Lecture 11

Learning Objectives
At the end of this class, students should be able to:

- understand the concept of curvature
- identify radius of curvature in various forms
- solve related problems

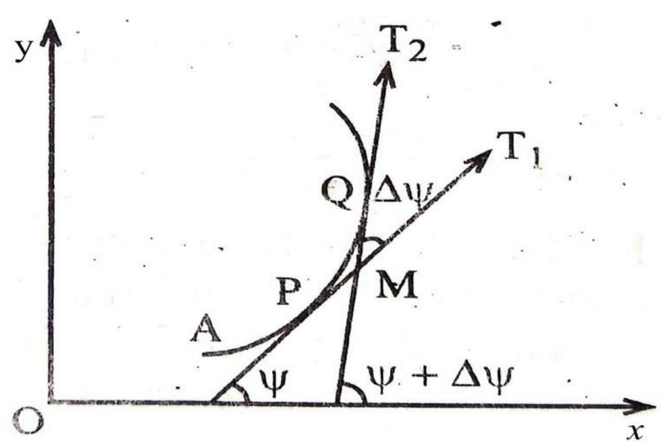
Curvature

The rate of change of direction of the curve with respect to arc is called curvature. It is denoted by κ.



Curvature at a Point

Let A be the fixed point from where we measure arc length. Let arc AP = s and arc AQ = s + Δs. Hence arc PQ = Δs. Let ψ and ψ+Δψ be the angles made by the tangents at P and Q respectively with x-axis. Then $\angle T_1MT_2 = \Delta\psi$.



The ratio $\frac{\Delta\psi}{\Delta s}$ is called average curvature of the arc PQ. The limiting value of $\frac{\Delta\psi}{\Delta s}$, as $\Delta s \to 0$, is called the curvature at point P.

Thus, $\lim_{\Delta s \to 0} \frac{\Delta\psi}{\Delta s} = \frac{d\psi}{ds} = \kappa$ which is the curvature at point P.

Radius of Curvature

The reciprocal of Curvature at any point P is called the radius of curvature at P and it is denoted by ρ. Thus $\rho = \frac{ds}{d\psi}$

Radius of Curvature in Cartesian Form

We know that $y_1 = \frac{dy}{dx} = \tan\psi$

Differentiating with respect to x, we get

$$y_2 = \sec^2\psi . \frac{d\psi}{dx}$$

$$= \sec^2\psi . \frac{d\psi}{ds} . \frac{ds}{dx}$$

$$= \sec^2\psi . \frac{d\psi}{ds} . \sec\psi \qquad \left[\because \frac{ds}{dx} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2} = \sqrt{1 + \tan^2\psi} = \sec\psi\right]$$

$$\therefore \quad \frac{ds}{d\psi} = \frac{\sec^3\psi}{y_2}$$

$$= \frac{(\sec^2\psi)^{3/2}}{y_2}$$

$$= \frac{(1+\tan^2\psi)^{3/2}}{y_2}$$

$$= \frac{(1+y_1^2)^{3/2}}{y_2}$$

Thus, $\rho = \frac{(1+y_1^2)^{3/2}}{y_2}$, where $y_1 = \frac{dy}{dx}$ and $y_2 = \frac{d^2y}{dx^2}$

This formula fails at the point on the curve if the tangent is parallel to y-axis.

In this case, we use

$$\rho = \frac{(1+x_1^2)^{3/2}}{x_2}, \text{ where } x_1 = \frac{dx}{dy} \text{ and } x_2 = \frac{d^2x}{dy^2}$$

Radius of Curvature in Polar Form

Let $x = r\cos\theta, y = r\sin\theta$. Then

$\frac{dx}{d\theta} = r_1\cos\theta - r\sin\theta, \frac{dy}{d\theta} = r_1\sin\theta + r\cos\theta$, where $r_1 = \frac{dr}{d\theta}$

Now, $y_1 = \frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{r_1\sin\theta + r\cos\theta}{r_1\cos\theta - r\sin\theta}$ and

$$y_2 = \frac{d^2y}{dx^2} = \frac{d}{d\theta}\left(\frac{r_1\sin\theta + r\cos\theta}{r_1\cos\theta - r\sin\theta}\right) . \frac{d\theta}{dx}$$

$$= \frac{r^2+2r_1^2-rr_2}{(r_1\cos\theta - r\sin\theta)^3}$$

Thus, $\rho = \frac{(r^2+r_1^2)^{3/2}}{r^2+2r_1^2-rr_2}$, where $r_2 = \frac{d^2r}{d\theta^2}$ $\left[\because \ \rho = \frac{(1+y_1^2)^{3/2}}{y_2}\right]$

Radius of Curvature in Parametric Form

Let $x = f(t), y = g(t)$. Then

Now, $y_1 = \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{g'(t)}{f'(t)}$ and

$$y_2 = \frac{d}{dx}\left\{\frac{g'(t)}{f'(t)}\right\} = \frac{d}{dt}\left\{\frac{g'(t)}{f'(t)}\right\}.\left(\frac{dt}{dx}\right)$$

$$= \frac{f'(t)g''(t)-g'(t)f''(t)}{\{f'(t)\}^3}$$

Thus, $\rho = \frac{[\{f'(t)\}^2+\{g'(t)\}^2]^{3/2}}{f'(t)g''(t)-g'(t)f''(t}$ $\left[\because \ \rho = \frac{(1+y_1^2)^{3/2}}{y_2}\right]$

Newton's Formula for Radius of Curvature at Origin

Let $y = f(x)$ be the equation of a given curve. By Maclaurin's theorem, its expansion is:

$$y = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + . \ . \ . \qquad \text{(i)}$$

The given curve passes through the origin so $f(0) = 0$. Suppose x-axis is the tangent line at the origin so $f'(0) = 0$. Then equation (i) becomes

$$y = \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + . \ . \ .$$

Dividing both sides by $\frac{x^2}{2!}$, we get

$$\frac{2y}{x^2} = f''(0) + \frac{x}{3}f'''(0) + . \ . \ .$$

Taking limit $x \to 0$ and consequently y also tends to 0, we have

$$\lim_{\substack{x\to 0\\ y\to 0}}\left(\frac{2y}{x^2}\right) = f''(0)$$

We know that $\rho = \frac{\left[1+\{f'(0)\}^2\right]^{3/2}}{f''(0)} = \frac{1}{\lim_{\substack{x\to 0\\ y\to 0}}\left(\frac{2y}{x^2}\right)}$

$$\therefore \qquad \rho = \lim_{\substack{x\to 0\\ y\to 0}}\left(\frac{x^2}{2y}\right)$$

Similarly, if a curve passes through the origin and y-axis is the tangent line. Then

$$\rho = \lim_{\substack{x \to 0 \\ y \to 0}} \left(\frac{y^2}{2x}\right)$$

Illustration

Find the radius of curvature at any point (s, ψ) of the curve: $s = a(e^{m\psi} - 1)$.

Solution

We have $s = a(e^{m\psi} - 1)$ then

$$\frac{ds}{d\psi} = ame^{m\psi}$$

We know that ρ = $\frac{ds}{d\psi}$

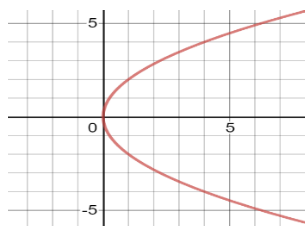
Thus, ρ $= ame^{m\psi}$

Illustration

Find the radius of curvature at any point of the curve: $y^2 = 4ax$.

Solution

We have $y^2 = 4ax$. The graph of this curve is as follows:



Since the tangent to this curve becomes parallel to y-axis, so applying $\rho = \frac{(1+x_1^2)^{3/2}}{x_2}$

Here, $x = \frac{y^2}{4a}$ then $\frac{dx}{dy} = \frac{y}{2a}$ and $\frac{d^2x}{dy^2} = \frac{1}{2a}$

∴ $\rho = \frac{[1+(y/2a)^2]^{3/2}}{(1/2a)}$

$$= 2a\left[1 + \frac{y^2}{4a^2}\right]^{3/2}$$

$$= \frac{2a[4a^2+y^2]^{3/2}}{[4a^2]^{3/2}}$$

$$= \frac{2a[4a^2+4xa]^{3/2}}{2^3 \times a^3}$$

$$= \frac{2a \times 2^3 \times a^{3/2}(a+x)^{3/2}}{2^3 \times a^3}$$

$$= \frac{2}{\sqrt{a}}(a + x)^{3/2}$$

Illustration

Find the radius of curvature at any point of the curve: $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

Solution

We have; $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. Differentiating with respect to *x*, we get

$$\frac{2x}{a^2} + \frac{2yy_1}{b^2} = 0$$

$$y_1 = -\frac{b^2}{a^2}\frac{x}{y}$$

Differentiating again, we get

$$y_2 = -\frac{b^2}{a^2}\left[\frac{y - xy_1}{y^2}\right]$$

$$= -\frac{\frac{b^2}{a^2}\left[y + x \times \frac{b^2 x}{a^2 y}\right]}{y^2}$$

$$= -\frac{b^4}{a^2y^3}\left[\frac{y^2}{b^2} + \frac{x^2}{a^2}\right]$$

$$= -\frac{b^4}{a^2y^3} \qquad \left[\because \ \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1\right]$$

Now, $\rho = \frac{\left(1+y_1^2\right)^{3/2}}{y_2}$

$$= \frac{\left(1+\frac{b^4x^2}{a^4y^2}\right)^{3/2}}{\left(-\frac{b^4}{a^2y^3}\right)}$$

$$= -\frac{\left(a^4y^2+b^4x^2\right)^{3/2}}{a^6y^3} \times \frac{a^2y^3}{b^4}$$

$$= -\frac{\left(a^4y^2+b^4x^2\right)^{3/2}}{a^4b^4}$$

Taking magnitude, we get

$$\rho = \frac{\left(a^4y^2+b^4x^2\right)^{3/2}}{a^4b^4}$$

Illustration

Find the radius of curvature at any point (r, θ) of the curve: $r^2 \cos 2\theta = a^2$.

Solution

We have $r^2 \cos 2\theta = a^2$. Differentiating with respect to θ, we get

$$-2r^2 \sin 2\theta + 2\cos 2\theta \,.\, r.\, r_1 = 0$$

or, $\quad r_1 = r \tan 2\theta$ and

$$r_2 = 2r\sec^2 2\theta + \tan 2\theta \,.\, r_1$$

$$= 2r\sec^2 2\theta + \tan 2\theta \,.\, r\tan 2\theta$$

$$= 2r\sec^2 2\theta + r\tan^2 2\theta$$

Now, $\rho = \frac{(r^2+r_1^2)^{3/2}}{r^2+2r_1^2-rr_2}$

$$= \frac{(r^2+r^2\tan^2 2\theta)^{3/2}}{r^2+2r^2\tan^2 2\theta-2r^2\sec^2 2\theta-r^2\tan^2 2\theta}$$

$$= \frac{r^3(\sec^2 2\theta)^{3/2}}{r^2(1+2\tan^2 2\theta-2\sec^2 2\theta)}$$

$$= \frac{r\sec^3 2\theta}{-\sec^2 2\theta}$$

$$= -r\sec 2\theta$$

$$= -\frac{r}{\cos 2\theta}$$

$$-\frac{r}{a^2/r^2} = -\frac{r^3}{a^2}$$

Taking magnitude, we get

$$\rho = \frac{r^3}{a^2}$$

Illustration

Find the radius of curvature at any point of the curve: $x = a(\cos t + t\sin t)$, $y = a(\sin t - t\cos t)$.

Solution

Here,

$$\frac{dx}{dt} = a(-\sin t + t\cos t + \sin t) = at\cos t$$

$$\frac{d^2x}{dt^2} = a\cos t - at\sin t$$

$$\frac{dy}{dt} = a(\cos t - \cos t + t\sin t) = at\sin t$$

$$\frac{d^2y}{dt^2} = at\cos t + a\sin t$$

Now

$$\rho = \frac{[\{f\prime(t)\}^2+\{g\prime(t)\}^2]^{3/2}}{f'(t)g''(t)-g'(t)f\prime\prime(t)}$$

$$= \frac{[\{at\cos t\}^2+\{at\sin t\}^2]^{3/2}}{at\cos t(at\cos t+a\sin t)-at\sin t(a\cos t-at\sin t)}$$

$$= \frac{(a^2t^2)^{3/2}}{a^2t^2\cos^2 t+a^2t\sin t\cos t-a^2t\sin t\cos t+a^2t^2\sin^2 t}$$

$= \frac{a^3t^3}{a^2t^2}$

$= at$

Illustration

Find the radius of curvature of the curve $3x^2 + 4x^3 - 12y = 0$ at origin.

Solution

We have $3x^2 + 4x^3 - 12y = 0$ …(i)

This curve passes through the origin and the tangent at the origin is given by $y = 0$ which is x-axis. [Equating the lowest term to zero]

So, $\rho = \lim_{\substack{x\to 0\\ y\to 0}} \left(\frac{x^2}{2y}\right)$

Now dividing both sides of equation (i) by 2y, we get

$$3\left(\frac{x^2}{2y}\right) + 4x\left(\frac{x^2}{2y}\right) - 6 = 0$$

Taking limit $x \to 0, y \to 0$, we have

$$3\lim_{\substack{x\to 0\\ y\to 0}} \left(\frac{x^2}{2y}\right) + 4\lim_{\substack{x\to 0\\ y\to 0}} x\left(\frac{x^2}{2y}\right) - 6 = 0$$

or, $3\rho + 4 \times 0 \times \rho - 6 = 0$

∴ $\rho = 2$

Exercise for Reader

1. Find the radius of curvature at any point (s, ψ) of the curve: $s = c \log \sec \psi$.
2. Find the radius of curvature at any point of the curve: $x^{2/3} + y^{2/3} = a^{2/3}$.
3. Find the radius of curvature at the point (2, 3) of the curve: $9x^2 + 4y^2 = 36x$.
4. Find the radius of curvature at any point (r, θ) of the curve: $r^2 = a^2 \cos 2\theta$.
5. Find the radius of curvature of the curve: $x = a\cos^3\theta, y = a\sin^3\theta$ at $\theta = \pi/4$.