Lecture 10

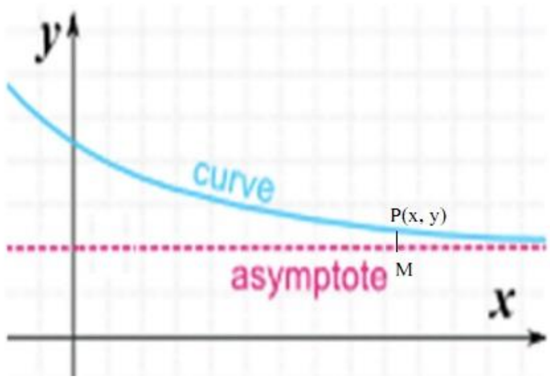
Learning Objectives

At the end of this class, students should be able to:

- understand the concept of asymptotes
- solve related problems

Asymptotes

A straight line is said to be an asymptote of a curve $y = f(x)$ if a perpendicular distance of the point $P(x, y)$ on the curve tends to zero as x or y or both tend to infinity.



Thus, an asymptote is a straight line that a curve approaches, as it heads towards infinity.

Horizontal Asymptote

The horizontal line $y = b$ is called a horizontal asymptote of the graph of $y = f(x)$ if either $\lim_{x\to\infty} f(x) = b$ or $\lim_{x\to-\infty} f(x) = b$.

The condition $\lim_{x\to\infty} f(x) = b$ is satisfied for the graph in the following figure (a).

Vertical Asymptote

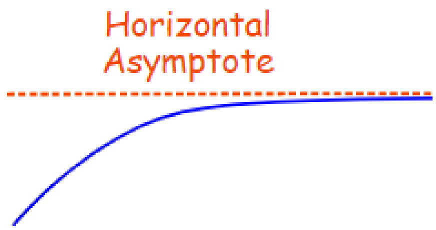
The vertical line $x = a$ is called a vertical asymptote of the graph of $y = f(x)$ if

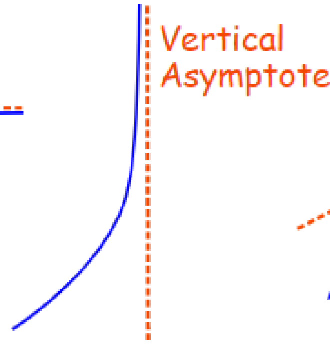
$$\lim_{x\to a^-} f(x) = \pm\infty \quad \text{or} \quad \lim_{x\to a^+} f(x) = \pm\infty$$

The condition $\lim_{x\to a^-} f(x) = \infty$ is satisfied for the graph in the following figure (b).

A graph can have an infinite number of vertical asymptotes but it can only have at most two horizontal asymptotes.

A graph will never touch a vertical asymptote; however, a graph may cross a horizontal asymptote.





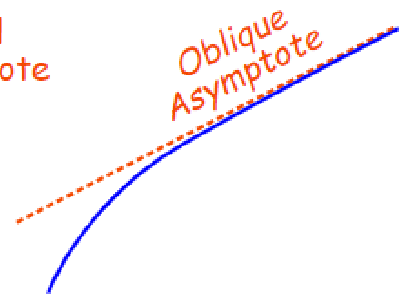


Figure (a) Figure (b) Figure (c)

Oblique Asymptote (Slanted-Line)

An asymptote which is neither parallel to x-axis nor parallel to y-axis is called an oblique asymptote. As x goes to infinity (or $-\infty$) then the curve goes towards a line $y = mx + c, m \neq 0$.

A rational function in which the degree of numerator is one greater than the degree of denominator has always an oblique asymptote.

Method of Finding Oblique Asymptotes of Algebraic Curves (Use of Taylor's Theorem)

Let the equation of curve $f(x, y) = 0$ of degree n can be written as:

$$(a_0x^n + a_1x^{n-1}y + a_2x^{n-2}y^2 + \ldots + a_ny^n) + (b_1x^{n-1} + b_2x^{n-2}y + \ldots + b_ny^{n-1}) + (c_2x^{n-2} + c_3x^{n-3}y + \ldots + c_ny^{n-2}) + \ldots = 0$$

where $a_0, a_1, a_2, \ \ldots, b_1, b_2, b_3, \ \ldots, c_2, c_3, c_4, \ \ldots$ are all constants.

Then the above equation can be written as

$$x^n\phi_n\left(\frac{y}{x}\right) + x^{n-1}\phi_{n-1}\left(\frac{y}{x}\right) + x^{n-2}\phi_{n-2}\left(\frac{y}{x}\right) + \ldots + x^r\phi_r\left(\frac{y}{x}\right) + \ldots = 0 \qquad \ldots (1)$$

Where $\phi_r\left(\frac{y}{x}\right)$ represents the algebraic polynomial in $\frac{y}{x}$ of degree r.

Let $y = mx + c$ … (2) be the asymptote of curve (1). Then from equation (1),

$$x^n\phi_n\left(m + \frac{c}{x}\right) + x^{n-1}\phi_{n-1}\left(m + \frac{c}{x}\right) + x^{n-2}\phi_{n-2}\left(m + \frac{c}{x}\right) + \ldots + x^r\phi_r\left(m + \frac{c}{x}\right) + \ldots = 0$$

By Taylor's theorem, we have

$$x^n\left\{\phi_n(m) + \frac{c}{x}\phi_n'(m) + \frac{1}{2!}\left(\frac{c}{x}\right)^2\phi_n''(m) + \ldots\right\} + x^{n-1}\left\{\phi_{n-1}(m) + \frac{c}{x}\phi_{n-1}'(m) + \frac{1}{2!}\left(\frac{c}{x}\right)^2\phi_{n-1}''(m) + \ldots\right\} + \ldots = 0$$

Arranging the terms in descending powers of x, we get

$$x^n\phi_n(m) + \{c\phi_n'(m) + \phi_{n-1}(m)\}x^{n-1}$$

$$+\left\{\frac{c^2}{2!}\phi_n''(m) + c\phi_{n-1}'(m) + \phi_{n-2}(m)\right\}x^{n-2} + \ . \ . \ . = 0$$

This is a polynomial equation of degree n in x. It has n roots: real or imaginary. If the line (2) is asymptote of the curve (1), then the line meets the curve at two points at infinity.

So, the coefficients of x^n and x^{n-1} must be zero.

i.e., $\phi_n(m) = 0$ … (3)

$c\phi_n'(m) + \phi_{n-1}(m) = 0$ … (4)

$\Rightarrow$ $c = -\frac{\phi_{n-1}(m)}{\phi_n'(m)}$; $\phi_n'(m) \neq 0$

Equation (3) is n^{th} degree in m, therefore it gives n values of m. To each value of m, we get the values of m whereas from (4), we get the corresponding value of c . Thus, for every pair of values of m and c, there will be an asymptote of the curve.

Note: If the equation $\phi_n(m) = 0$ has two equal roots, then the value of $\phi_n'(m)$ becomes zero. In this case, $c = -\frac{\phi_{n-1}(m)}{\phi_n'(m)}$ tends to infinity, provided $\phi_{n-1}(m) \neq 0$. In such a case, there will be no asymptote.

Moreover, if $\phi_{n-1}(m)$ is also zero, then the value of c can be obtained by making the coefficient of x^{n-2}zero. i.e., $\frac{c^2}{2!}\phi_n''(m) + c\phi_{n-1}'(m) + \phi_{n-2}(m) = 0$.

If the equation $\phi_n(m) = 0$ has three equal roots, then the value of c can be obtained by making the coefficient of x^{n-3} zero. i.e., $\frac{c^3}{3!}\phi_n'''(m) + \frac{c^2}{2!}\phi_{n-1}''(m) + c\phi_{n-2}'(m) + \phi_{n-3}(m) = 0$.

Procedure for Finding Asymptotes of curve $f(x, y) = 0$

1. If the degree of equation = degree of x that involve in equation, then there is no asymptote parallel to x-axis.
   a. If the degree of x is less than the degree of equation, there may exist horizontal asymptotes.
   b. To find horizontal asymptotes, set the coefficient of highest power of x equal to zero.
2. If the degree of equation = degree of y that involve in equation, then there is no vertical asymptote.
   a. If the degree of y is less than the degree of equation, there may exist vertical asymptotes.
   b. To find vertical asymptotes, set the coefficient of highest power of y equal to zero.
3. To find oblique asymptotes of the form $y = mx + c$; first, we put $x = 1$ and $y = m$ in n^{th} degree terms of the given equation to get $\phi_n(m)$ and $(n - 1)^{th}$ degree terms of the given equation to get $\phi_{n-1}(m)$, and so on. After that, we solve the equation $\phi_n(m) = 0$ to get the values of m. There could be the following situations:
   a. If the roots the equation $\phi_n(m) = 0$ are m_1 and m_2 which are real and different, then the value of c can be found by $c = -\frac{\phi_{n-1}(m)}{\phi_n'(m)}$.

b. If the equation $\phi_n(m) = 0$ has two equal roots, then the value of c can be found by solving the equation: $\frac{c^2}{2!}\phi_n''(m) + c\phi_{n-1}'(m) + \phi_{n-2}(m) = 0$.

c. If the equation $\phi_n(m) = 0$ has three equal roots, then the value of c can be found by solving the equation: $\frac{c^3}{3!}\phi_n'''(m) + \frac{c^2}{2!}\phi_{n-1}''(m) + c\phi_{n-2}'(m) + \phi_{n-3}(m) = 0$.

Illustration

Find the asymptotes of the curve: $y^3 - x^2y + 2y^2 + 4y + x = 0$.

Solution

The given curve is of degree 3, so there exists at most three asymptotes.

Since y^3 is present in the equation, so there is no asymptote parallel to y-axis.

Since there is no x^3, so there are asymptotes parallel to x-axis. To find horizontal asymptotes, we equate the coefficient of highest degree term in *x* to zero.

i.e., $\quad -y = 0$

$\Rightarrow \quad y = 0$

Thus, the horizontal asymptote of the given curve is given by $y = 0$.

Now, we find oblique asymptotes which will be of the form $y = mx + c$.

Putting $x = 1$ and $y = m$ in the third-degree terms of the given equation to get $\phi_3(m)$,

$$\phi_3(m) = m^3 - m$$

Similarly, putting $x = 1$ and $y = m$ in the second-degree terms of the given equation to get $\phi_2(m)$,

$$\phi_2(m) = 2m^2$$

Also, $\phi_3'(m) = 3m^2 - 1$

Now, solving the equation $\phi_3(m) = 0$ to get the values of *m*.

i.e., $\quad m^3 - m = 0$

or, $\quad m(m-1)(m+1) = 0$

$\therefore \quad m = 0, 1, -1$

Now, the value of c can be obtained by using

$$c = -\frac{\phi_{n-1}(m)}{\phi_n'(m)} = -\frac{\phi_2(m)}{\phi_3'(m)} = -\frac{2m^2}{3m^2-1}$$

At $m = 0, \quad c = 0$

At $m = 1, \quad c = -1$

At $m = -1, \quad c = -1$

Thus, the asymptotes of the curve are:

$$y = 0, y = x - 1 \text{ and } y = -x - 1$$

Illustration

Find the asymptotes of the curve: $y = \frac{1}{(x+2)^2}$.

Solution

We have $(x + 2)^2 y = 1$

or, $\quad x^2 y + 4xy + 4y - 1 = 0$

The given curve is of degree 3, so it has at most three asymptotes.

There are no terms involving $x^3$and y^3. So, there are asymptotes parallel to x-axis and y-axis.

To find the asymptotes parallel to x-axis, we equate the coefficient of highest degree term in *x* to zero.

i.e., $\quad y = 0$

Thus, the horizontal asymptote of the given curve is given by $y = 0$.

To find the asymptotes parallel to y-axis, we equate the coefficient of highest degree term in *y* to zero.

i.e., $\quad x^2 + 4x + 4 = 0$

or, $\quad (x + 2)(x + 2) = 0$

Thus, the vertical asymptote of the given curve is given by $x = -2$.

Now, we find oblique asymptotes which will be of the form $y = mx + c$.

Putting $x = 1$ and $y = m$ in the third-degree terms and second-degree terms of the given equation, we get

$$\phi_3(m) = m \text{ and } \phi_2(m) = 4m$$

Also, $\phi_3'(m) = 1$

Now, solving the equation $\phi_3(m) = 0$ to get the values of *m*.

i.e., $\quad m = 0$

Since $m = 0$, so there is no oblique asymptote.

Illustration

Find the asymptotes of the curve: $x^2(x - y)^2 - a^2(x^2 + y^2) = 0$.

Solution

We have $x^2(x - y)^2 - a^2(x^2 + y^2) = 0$

or, $\quad x^4 - 2x^3 y + x^2 y^2 - a^2 x^2 - a^2 y^2 = 0$

The given curve is of degree 4, so it has at most four asymptotes.

Since x^4 is present in the equation, so there is no asymptote parallel to x-axis.

There are no terms involving y^4. So, there are asymptotes parallel to y-axis.

To find the asymptotes parallel to y-axis, we equate the coefficient of highest degree term in y to zero.

i.e., $x^2 - a^2 = 0$

or, $(x-a)(x+a) = 0$

Thus, the vertical asymptotes of the given curve are: $x = -a$ and $x = a$.

Now, we find oblique asymptotes which will be of the form $y = mx + c$.

Putting $x = 1$ and $y = m$ in the fourth-degree terms and third-degree terms of the given equation, we get

$$\phi_4(m) = 1 - 2m + m^2 \text{ and } \phi_3(m) = 0$$

Now, solving the equation $\phi_4(m) = 0$ to get the values of m.

i.e., $1 - 2m + m^2 = 0$

or, $(m-1)(m-1) = 0$

∴ $m = 1, 1$ (Repeated)

Since two roots of the equation $\phi_4(m) = 0$ are equal, therefore the value of c can be obtained by using

$$\frac{c^2}{2!}\phi_n''(m) + c\phi_{n-1}'(m) + \phi_{n-2}(m) = 0$$

i.e., $\frac{c^2}{2!}\phi_4''(m) + c\phi_3'(m) + \phi_2(m) = 0$

or, $\frac{c^2}{2} \times 2 + c \times 0 - a^2 - a^2m^2 = 0$ $[\because\ \phi_4''(m) = 2, \phi_3'(m) = 0, \phi_2(m) = -a^2 - a^2m^2]$

or, $c^2 - 2a^2 = 0$ $[\because m = 1]$

∴ $c = \pm\sqrt{2}a$

Thus, the oblique asymptotes are $y = x \pm \sqrt{2}a$

Hence, the asymptotes of the given curve are: $x = \pm a, y = x \pm \sqrt{2}a$.

Illustration

Find the asymptotes of the curve: $y = e^{1/x}$.

Solution

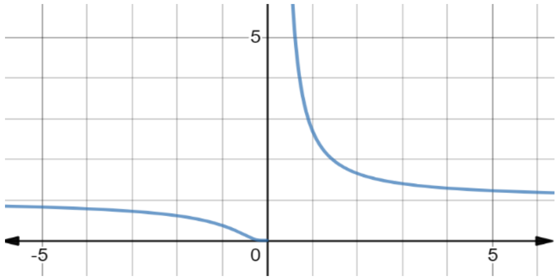
For horizontal asymptotes;

$$\lim_{x\to+\infty} e^{1/x} = e^0 = 1$$

and $\lim_{x\to-\infty} e^{1/x} = e^0 = 1$

Therefore, the function $y = e^{1/x}$ has a horizontal asymptote $y = 1$.

We have $y = e^{1/x}$. The graph of this function is as follows.



For vertical asymptotes;

When x approach zero from right hand side, $1/x$ is closer to $+\infty$ and thus the limiting value of $e^{1/x}$ is $+\infty$.

i.e., $\lim_{x \to 0^+} e^{1/x} = +\infty$

When x approach zero from left hand side, $1/x$ is closer to $-\infty$ and thus the limiting value of $e^{1/x}$ is 0.

i.e., $\lim_{x \to 0^-} e^{1/x} = 0$

Thus, the function $y = e^{1/x}$ has vertical asymptote $x = 0$ when x is greater than 0 but there is no vertical asymptote when x is less than 0.
Thus, the asymptotes of the given function are $x = 0$ and $y = 1$.

Exercise for Reader

Find the asymptotes of the following curves:

1. $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$
2. $x^2 - y^2 - 6x + 4y + 1 = 0$
3. $x^2 y - 1 = 0$
4. $x^3 + y^3 = 3xy$
5. $x^3 + 3x^2 y - xy^2 - 3y^3 + x^2 - 2xy + 3y^2 + 4x + 5 = 0$
6. $(x^2 - y^2)^2 - 8(x^2 + y^2) + 8x - 16 = 0$