Lecture 9

Learning Objectives
At the end of this class, students should be able to:

- apply L'Hôpital's Rule
- solve related problems

Indeterminate Form $\infty - \infty$

If $\lim_{x\to a} f(x) = \infty$ and $\lim_{x\to a} g(x) = \infty$ then $\lim_{x\to a}[f(x) - g(x)]$ will be of the form $\infty - \infty$.

The form $\infty - \infty$ can be reduced to $\frac{0}{0}$ by writing

$$\lim_{x\to a}[f(x) - g(x)] \qquad = \lim_{x\to a}\left[\frac{\frac{1}{g(x)} - \frac{1}{f(x)}}{\frac{1}{f(x).g(x)}}\right]$$

Illustration

Evaluate: $\lim_{x\to 0}\left[\frac{1}{x^2} - \cot^2 x\right]$

Solution

We have $\lim_{x\to 0}\left[\frac{1}{x^2} - \cot^2 x\right]$ which is of the form $\infty - \infty$.

Now, $\lim_{x\to 0}\left[\frac{1}{x^2} - \cot^2 x\right] \qquad = \lim_{x\to 0}\left[\frac{1}{x^2} - \frac{1}{\tan^2 x}\right]$

$$= \lim_{x\to 0}\left[\frac{\tan^2 x - x^2}{x^2 \tan^2 x}\right]$$

$$= \lim_{x\to 0}\left[\frac{\tan^2 x - x^2}{x^4}\right] \times \frac{1}{\lim_{x\to 0}\left[\frac{\tan x}{x}\right]^2}$$

$$= \lim_{x\to 0}\left[\frac{\tan^2 x - x^2}{x^4}\right] \qquad \left[\frac{0}{0} \text{ form; applying L'Hôpital's Rule}\right]$$

$$= \lim_{x\to 0}\left[\frac{2\tan x.\ \sec^2 x - 2x}{4x^3}\right]$$

$$= \lim_{x\to 0}\left[\frac{2\tan x(1 + \tan^2 x) - 2x}{4x^3}\right]$$

$$= \lim_{x\to 0}\left[\frac{2\tan x + 2\tan^3 x - 2x}{4x^3}\right] \qquad \left[\frac{0}{0} \text{ form; applying L'Hôpital's Rule}\right]$$

$$= \lim_{x \to 0} \left[\frac{2\sec^2 x + 6\tan^2 x . \sec^2 x - 2}{12x^2} \right]$$

$$= \lim_{x \to 0} \left[\frac{2\tan^2 x + 6\tan^2 x . \sec^2 x}{12x^2} \right]$$

$$= \lim_{x \to 0} \left[\frac{\tan x}{x} \right]^2 \times \lim_{x \to 0} \left[\frac{2 + 6\ \sec^2 x}{12} \right]$$

$$= \lim_{x \to 0} \left[\frac{2 + 6\ \sec^2 x}{12} \right]$$

$$= \frac{2+6}{12} = \frac{8}{12} = \frac{2}{3}$$

Indeterminate Forms $0^0, \infty^0, 1^\infty$

To determine $\lim_{x \to a} [f(x)]^{g(x)}$, when

i) $\lim_{x \to a} f(x) = 0$ and $\lim_{x \to a} g(x) = 0$.

ii) $\lim_{x \to a} f(x) = \infty$ and $\lim_{x \to a} g(x) = 0$.

iii) $\lim_{x \to a} f(x) = 1$ and $\lim_{x \to a} g(x) = \infty$.

We adapt the following steps:

1. We write $y = [f(x)]^{g(x)}$ so that $\ln y = g(x). \ln f(x)$.
2. We take limit on both sides,

$$\lim_{x \to a} [\ln y] = \lim_{x \to a} [g(x). \ln f(x)]$$

In each of the three cases, we see that the right-hand side assumes the indeterminate form $0 \times \infty$. Which can be reduced either to the form $\frac{0}{0}$ or to the form $\frac{\infty}{\infty}$.

3. We apply L'Hôpital's Rule
4. If we let $\lim_{x \to a} [g(x). \ln f(x)] = L$ then above expression becomes $\lim_{x \to a} [\ln y] = L$. Using the fact $\lim_{x \to a} [\ln y] = \ln \left[\lim_{x \to a} (y) \right]$; this expression becomes

$$\ln \left[\lim_{x \to a} (y) \right] = L \qquad \Rightarrow \lim_{x \to a} (y) = e^L$$

Thus, $\lim_{x \to a} [f(x)]^{g(x)} = e^L$

Illustration

Evaluate: $\lim_{x \to 0^+} x^{2 \sin x}$

Solution

We have, $\lim_{x \to 0^+} x^{2\sin x}$ which is of the form 0^0.

Let $y = x^{2\sin x}$

Taking natural logarithm on both sides, we get

$$\ln y = (2\sin x)\ln x$$

Taking limit on both sides, we get

$$\lim_{x \to 0^+}[\ln y] = \lim_{x \to 0^+}[(2\sin x)\ln x] \qquad [0 \times \infty \text{ form}]$$

$$= \lim_{x \to 0^+} \frac{\ln x}{2\,\text{cosec}\, x} \qquad \left[\frac{\infty}{\infty} \text{ form; applying L'Hôpital's Rule}\right]$$

$$= \lim_{x \to 0^+} \frac{1/x}{2\,(-\text{cosec}\, x \cot x)}$$

$$= \lim_{x \to 0^+} \frac{-\sin^2 x}{2x\cos x} \qquad \left[\frac{0}{0} \text{ form; applying L'Hôpital's Rule}\right]$$

$$= \lim_{x \to 0^+} \frac{-2\sin x\cos x}{2\cos x + 2x\sin x}$$

$$= \frac{-2 \times 0}{2 \times 1 + 2 \times 0} \qquad = 0$$

Hence, $\lim_{x \to 0^+}[\ln y] = 0$

or, $\ln\left[\lim_{x \to 0^+}(y)\right] = 0$

or, $\lim_{x \to 0^+}(y) = e^0$

$\therefore$ $\lim_{x \to 0^+} x^{2\sin x} = 1$

Illustration

Evaluate: $\lim_{x \to 0}(\cos x)^{\cot^2 x}$

Solution

We have, $\lim_{x \to 0}(\cos x)^{\cot^2 x}$ which is of the form 1^∞.

Let $y = (\cos x)^{\cot^2 x}$

Taking natural logarithm on both sides, we get

$$\ln y = \cot^2 x \ln(\cos x)$$

Taking limit on both sides, we get

$$\lim_{x \to a}[\ln y] = \lim_{x \to 0}[\cot^2 x \ln(\cos x)] \qquad [0 \times \infty \text{ form}]$$

$$= \lim_{x \to 0}\left[\frac{\ln(\cos x)}{\tan^2 x}\right] \qquad \left[\frac{0}{0} \text{ form; applying L'Hôpital's Rule}\right]$$

$$= \lim_{x \to 0}\left[\frac{1}{\cos x}\frac{(-\sin x)}{2\tan x.\sec^2 x}\right]$$

$$= \lim_{x \to 0}\left[-\frac{\tan x}{2\tan x.\sec^2 x}\right]$$

$$= \lim_{x \to 0}\left[-\frac{1}{2\sec^2 x}\right]$$

$$= -1/2$$

Hence, $\lim_{x \to a}[\ln y] = -\frac{1}{2}$

or, $\ln\left[\lim_{x \to a}(y)\right] = -\frac{1}{2}$

or, $\lim_{x \to a}(y) = e^{-\frac{1}{2}}$

$\therefore$ $\lim_{x \to 0}(\cos x)^{\cot^2 x} = e^{-\frac{1}{2}}$

Illustration

Show that $\lim_{x \to \infty}\left[x - x^2 \log\left(1 + \frac{1}{x}\right)\right]$

Solution

We have, $\lim_{x \to \infty}\left[x - x^2 \log\left(1 + \frac{1}{x}\right)\right]$

Putting $x = \frac{1}{y}$, when $x \to \infty$ then $y \to 0$. Then

$$\lim_{x\to\infty}\left[x - x^2\log\left(1+\frac{1}{x}\right)\right] = \lim_{y\to 0}\left[\frac{1}{y} - \frac{1}{y^2}\log(1+y)\right]$$

$$= \lim_{y\to 0}\left[\frac{y-\log(1+y)}{y^2}\right] \qquad \left[\frac{0}{0}\text{ form; applying L'Hôpital's Rule}\right]$$

$$= \lim_{y\to 0}\left[\frac{1-\frac{1}{1+y}}{2y}\right] \qquad \left[\frac{0}{0}\text{ form; applying L'Hôpital's Rule again}\right]$$

$$= \lim_{y\to 0}\left[\frac{\frac{1}{(1+y)^2}}{2}\right]$$

$$= \frac{\frac{1}{(1+0)^2}}{2} = \frac{1}{2}$$

Illustration

If $\lim_{x\to 0}\frac{a\sin x - \sin 2x}{\tan^3 x}$ is finite, show that $a = 2$ and limit is 1.

Solution

We have, $\lim_{x\to 0}\frac{a\sin x - \sin 2x}{\tan^3 x}$

$$= \lim_{x\to 0}\frac{a\sin x - \sin 2x}{x^3} \times \frac{1}{\lim_{x\to 0}\left(\frac{\tan x}{x}\right)^3}$$

$$= \lim_{x\to 0}\frac{a\sin x - \sin 2x}{x^3} \qquad \left[\frac{0}{0}\text{ form; applying L'Hôpital's Rule}\right]$$

$$= \lim_{x\to 0}\frac{a\cos x - 2\cos 2x}{3x^2}$$

When $x \to 0$, the denominator $3x^2$ tends to zero. So, the expression $\frac{a\cos x - 2\cos 2x}{3x^2}$ will not tend to a finite number unless $(a\cos x - 2\cos 2x) \to 0$ as $x \to 0$.

This requires $a - 2 = 0 \quad \Rightarrow a = 2$.

When $a = 2$,

$$\lim_{x\to 0}\frac{a\cos x - 2\cos 2x}{3x^2} = \lim_{x\to 0}\frac{2\cos x - 2\cos 2x}{3x^2} \qquad \left[\frac{0}{0}\text{ form; applying L'Hôpital's Rule}\right]$$

$$= \lim_{x\to 0}\frac{-2\sin x - 2\sin 2x}{6x} \qquad \left[\frac{0}{0}\text{ form; applying L'Hôpital's Rule again}\right]$$

$$= \lim_{x \to 0} \frac{-2\cos x + 4\cos 2x}{6}$$

$$= \frac{-2\times 1 + 8\times 1}{6} = 1$$

Exercise for Reader

1. Evaluate: $\lim_{x \to 1^+} \left(\frac{1}{x-1} - \frac{1}{\ln x} \right)$
2. Evaluate: $\lim_{x \to 0} x^x$
3. Evaluate: $\lim_{x \to 1^+} x^{1/(x-1)}$
4. Evaluate: $\lim_{x \to 0} (\cot x)^{\sin 2x}$
5. Evaluate: $\lim_{x \to \infty} \left(1 + \frac{1}{x}\right)^x$
6. Show that $\lim_{x \to 0} \left[\frac{x\cos x - \log(1+x)}{x^2} \right] = \frac{1}{2}$