Lecture 8

Learning Objectives
At the end of this class, students should be able to:

- apply L'Hôpital's Rule
- solve related problems

Indeterminate Forms

If the functions $f(x)$ and $g(x)$ both tend to zero as $x \to a$, then $\lim_{x\to a} \frac{f(x)}{g(x)}$ is of the form $\frac{0}{0}$ which has no meaning in mathematics. The form $\frac{0}{0}$ is called an indeterminate form.

The other indeterminate forms are: $\frac{\infty}{\infty},\ 0 \times \infty,\ \infty - \infty,\ 0^0,\ 1^\infty,$ and ∞^0.

Indeterminate Form $\frac{0}{0}$

Let us know the L'Hôpital's Rule to handle the indeterminate form $\frac{0}{0}$ while evaluating the limit.

L'Hôpital's Rule

If the functions $f(x)$ and $g(x)$ as also their derivatives $f'(x)$ and $g'(x)$ are continuous at $x = a$ and $f(a) = g(a) = 0$ then

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$$

$$= \frac{f'(a)}{g'(a)}; \text{ provided that } g'(a) \neq 0.$$

Generalization

If $f'(a)$ and $g'(a)$ are both zero, then

$$\lim_{x\to a} \frac{f'(x)}{g'(x)} = \lim_{x\to a} \frac{f''(x)}{g''(x)} = \frac{f''(a)}{g''(a)}\text{ ; provided that } g''(a) \neq 0.$$

and so on.

Illustration

Evaluate: $\lim_{x\to 0} \frac{x - \sin^{-1} x}{\sin^3 x}$

Solution

We have $\lim_{x\to 0}\frac{x-\sin^{-1}x}{\sin^3 x}$

$$= \lim_{x\to 0}\frac{x-\sin^{-1}x}{x^3}\times\frac{1}{\lim_{x\to 0}\left(\frac{\sin x}{x}\right)^3}$$

$$= \lim_{x\to 0}\frac{x-\sin^{-1}x}{x^3} \qquad \left[\frac{0}{0}\text{ form; applying L'Hôpital's Rule}\right] \qquad \left[\because \lim_{x\to 0}\frac{\sin x}{x}=1\right]$$

$$= \lim_{x\to 0}\frac{1-\frac{1}{\sqrt{1-x^2}}}{3x^2} \qquad \left[\frac{0}{0}\text{ form; applying L'Hôpital's Rule again}\right]$$

$$= \lim_{x\to 0}\frac{(1/2)(1-x^2)^{-\frac{3}{2}}\times(-2x)}{6x}$$

$$= \lim_{x\to 0}-\frac{1}{6}(1-x^2)^{-\frac{3}{2}} \qquad = -\frac{1}{6}$$

Illustration

Evaluate: $\lim_{x\to 0}\frac{\cosh x-\cos x}{x\sin x}$

Solution

We have $\lim_{x\to 0}\frac{\cosh x-\cos x}{x\sin x}$

$$= \lim_{x\to 0}\left(\frac{\cosh x-\cos x}{x^2}\right)\times\lim_{x\to 0}\left(\frac{x}{\sin x}\right)$$

$$= \lim_{x\to 0}\left(\frac{\cosh x-\cos x}{x^2}\right) \qquad \left[\frac{0}{0}\text{ form; applying L'Hôpital's Rule}\right] \qquad \left[\because \lim_{x\to 0}\frac{\sin x}{x}=1\right]$$

$$= \lim_{x\to 0}\left(\frac{\sinh x-\sin x}{2x}\right) \qquad \left[\frac{0}{0}\text{ form; applying L'Hôpital's Rule again}\right]$$

$$= \lim_{x\to 0}\left(\frac{\cosh x+\cos x}{2}\right)$$

$$= \frac{1+1}{2} \quad = 1$$

Indeterminate Form $\frac{\infty}{\infty}$

Theorem: If $\lim_{x\to a}f(x)=\infty$ and $\lim_{x\to a}g(x)=\infty$ then $\lim_{x\to a}\frac{f(x)}{g(x)}=\lim_{x\to a}\frac{f'(x)}{g'(x)}$. This theorem is also true when $x\to\infty$ instead of $x\to a$.

Illustration

Evaluate: $\lim\limits_{x \to 0^+} \frac{\log(\sin x)}{\cot x}$

Solution

We have $\lim\limits_{x \to 0^+} \frac{\log(\sin x)}{\cot x}$ $\left[\frac{\infty}{\infty} \text{ form; applying L'Hôpital's Rule}\right]$

$= \lim\limits_{x \to 0^+} \frac{\frac{\cos x}{\sin x}}{-\operatorname{cosec}^2 x}$

$= \lim\limits_{x \to 0^+} (-\cos x \sin x)$

$= -1 \times 0 = 0$

Illustration

Evaluate: $\lim\limits_{x \to \infty} \frac{x^4}{e^x}$

Solution

We have $\lim\limits_{x \to \infty} \frac{x^4}{e^x}$ $\left[\frac{\infty}{\infty} \text{ form; applying L'Hôpital's Rule}\right]$

$= \lim\limits_{x \to \infty} \frac{4x^3}{e^x}$ $\left[\frac{\infty}{\infty} \text{ form; applying L'Hôpital's Rule again}\right]$

$= \lim\limits_{x \to \infty} \frac{12x^2}{e^x}$ $\left[\frac{\infty}{\infty} \text{ form; applying L'Hôpital's Rule again}\right]$

$= \lim\limits_{x \to \infty} \frac{24x}{e^x}$ $\left[\frac{\infty}{\infty} \text{ form; applying L'Hôpital's Rule again}\right]$

$= \lim\limits_{x \to \infty} \frac{24}{e^x}$

$= \frac{24}{e^\infty}$

$= \frac{24}{\infty} = 0$

Illustration

Evaluate: $\lim\limits_{x \to 0^+} \log_{\tan x}(\tan 2x)$

Solution

We have $\lim_{x\to 0^+} \log_{\tan x}(\tan 2x)$

$$= \lim_{x\to 0^+} \frac{\log(\tan 2x)}{\log(\tan x)} \qquad \left[\frac{\infty}{\infty} \text{ form; applying L'Hôpital's Rule}\right]$$

$$= \lim_{x\to 0^+} \frac{\left(\frac{2\sec^2 2x}{\tan 2x}\right)}{\left(\frac{\sec^2 x}{\tan x}\right)}$$

$$= \lim_{x\to 0^+} \frac{\left(\frac{2}{\cos^2 2x}\times\frac{\cos 2x}{\sin 2x}\right)}{\left(\frac{1}{\cos^2 x}\times\frac{\cos x}{\sin x}\right)}$$

$$= \lim_{x\to 0^+} \left(\frac{2\sin x\cos x}{\cos 2x\sin 2x}\right)$$

$$= \lim_{x\to 0^+} \left(\frac{1}{\cos 2x}\right)$$

$$= \frac{1}{1} = 1$$

Illustration

Evaluate: $\lim_{x\to\infty} \frac{x^n}{e^x}$

Solution

We have $\lim_{x\to\infty} \frac{x^n}{e^x}$ $\qquad \left[\frac{\infty}{\infty} \text{ form; applying L'Hôpital's Rule}\right]$

$$= \lim_{x\to\infty} \frac{nx^{n-1}}{e^x} \qquad \left[\frac{\infty}{\infty} \text{ form; applying L'Hôpital's Rule again}\right]$$

$$= \lim_{x\to\infty} \frac{n(n-1)x^{n-2}}{e^x} \qquad \left[\frac{\infty}{\infty} \text{ form; applying L'Hôpital's Rule again}\right]$$

Continuing the process up to n^{th} times, then

$$= \lim_{x\to\infty} \frac{n(n-1)(n-2)\,.\,.\,.\,2.1.x^0}{e^x}$$

$$= \lim_{x\to\infty} \frac{n!}{e^x}$$

$$= \frac{n!}{e^\infty} = \frac{n!}{\infty} = 0$$

Indeterminate Form 0 $\times \infty$

When $\lim_{x \to a} f(x) = 0$ and $\lim_{x \to a} g(x) = \infty$ then $\lim_{x \to a}[f(x).g(x)]$ will be of the form $0 \times \infty$.

The above limit can be written as

$$\lim_{x \to a}[f(x).g(x)] \quad = \lim_{x \to a}\left[\frac{f(x)}{1/g(x)}\right] \text{ or } \lim_{x \to a}\left[\frac{g(x)}{1/f(x)}\right]$$

Which is of the form $\frac{0}{0}$ or of the form $\frac{\infty}{\infty}$. It can be easily evaluated by L'Hôpital's Rule. While choosing the numerator ILATE rule can be used.

Illustration

Evaluate: $\lim_{x \to 0^+} x \ln x$

Solution

We have $\lim_{x \to 0^+} x \ln x$ which is of the form $0 \times \infty$.

Now, $\lim_{x \to 0^+} x \ln x \quad = \lim_{x \to 0^+} \frac{\ln x}{1/x} \quad \left[\frac{\infty}{\infty} \text{ form; applying L'Hôpital's Rule}\right]$

$$= \lim_{x \to 0^+} \frac{1/x}{-1/x^2}$$

$$= \lim_{x \to 0^+}(-x) = 0$$

Illustration

Evaluate: $\lim_{x \to a}\left[(a - x).\tan\left(\frac{\pi x}{2a}\right)\right]$

Solution

We have $\lim_{x \to a}\left[(a - x).\tan\left(\frac{\pi x}{2a}\right)\right]$ which is of the form $0 \times \infty$.

Now, $\lim_{x \to a}\left[(a - x).\tan\left(\frac{\pi x}{2a}\right)\right] \quad = \lim_{x \to a}\left[\frac{(a-x)}{\cot\left(\frac{\pi x}{2a}\right)}\right] \quad \left[\frac{0}{0} \text{ form; applying L'Hôpital's Rule}\right]$

$$= \lim_{x \to a} \frac{(-1)}{\left(-\frac{\pi}{2a}\right)\operatorname{cosec}^2\left(\frac{\pi x}{2a}\right)}$$

$$= \frac{(-1)}{\left(-\frac{\pi}{2a}\right) \times 1} = \frac{2a}{\pi}$$

Illustration

Evaluate: $\lim_{x \to 0^+} x^m (\log x)^n$

Solution

We have, $\lim_{x \to 0^+} x^m (\log x)^n$ which is of the form $0 \times \infty$.

Now, $\lim_{x \to 0^+} x^m (\log x)^n = \lim_{x \to 0^+} \frac{(\log x)^n}{(1/x^m)}$ $\left[\frac{\infty}{\infty} \text{ form; applying L'Hôpital's Rule}\right]$

$$= \lim_{x \to 0^+} \frac{n(\log x)^{n-1}.(1/x)}{-mx^{-m-1}}$$

$$= \lim_{x \to 0^+} \left[-\frac{n}{m} \frac{(\log x)^{n-1}}{x^{-m}}\right] \qquad \left[\frac{\infty}{\infty} \text{ form; applying L'Hôpital's Rule}\right]$$

$$= \lim_{x \to 0^+} \left[-\frac{n}{m} \frac{(n-1)(\log x)^{n-2}.(1/x)}{-mx^{-m-1}}\right]$$

$$= \lim_{x \to 0^+} \left[(-1)^2 \frac{n(n-1)}{m^2} \frac{(\log x)^{n-2}}{x^{-m}}\right] \qquad \left[\frac{\infty}{\infty} \text{ form}\right]$$

Applying L'Hôpital's Rule (n – 2) times, we get

$$= \lim_{x \to 0^+} \left[(-1)^n \frac{n!}{m^n} \frac{(\log x)^0}{x^{-m}}\right]$$

$$= (-1)^n \frac{n!}{m^n \times \infty} = 0$$

Exercise for Reader

1. Evaluate: $\lim_{x \to 0} \frac{x^n - a^n}{x - a}$
2. Evaluate: $\lim_{x \to 0} \frac{x - \sin x}{x^3}$
3. Evaluate: $\lim_{x \to \infty} \frac{x^3}{e^x}$
4. Evaluate: $\lim_{x \to 0^+} \frac{(\ln x)^2}{\ln(\sin x)}$
5. Evaluate: $\lim_{x \to 0} x^2 \ln x$