Lecture 7

Learning Objectives

At the end of this class, students should be able to:

- understand the concept of Lagrange's Mean Value theorem
- understand the concept of Cauchy's Mean Value theorem
- solve related problems

Lagrange's Mean Value Theorem

Suppose $y = f(x)$ be a continuous function over a closed interval $[a, b]$ and differentiable in open interval (a, b) then there exists at least one c in (a, b) at which $\frac{f(b)-f(a)}{b-a} = f'(c)$.

Proof: Let us draw a line through the point $A(a, f(a))$ and $B(b, f(b))$ as shown in the following figure (b).

Using equation of straight line in point slope form: $y - y_1 = m(x - x_1)$, the secant line (AB) is the graph of the function

$$g(x) = f(a) + \frac{f(b)-f(a)}{b-a}(x - a) \qquad (1)$$

As shown in the following graph (c), the vertical difference between the graphs of f and g at x is

$$h(x) = f(x) - g(x).$$

Using equation (1), we get

$$h(x) = f(x) - f(a) - \frac{f(b)-f(a)}{b-a}(x - a) \qquad (2)$$

The function $h(x)$ satisfies the hypotheses of Rolle's Theorem on interval $[a, b]$. It is continuous on closed interval $[a, b]$ and differentiable on open interval (a, b). Also $h(a) = h(b) = 0$. Therefore, $h'(c) = 0$ at some point c in (a, b).

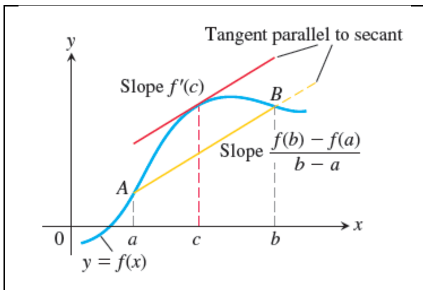

Figure (a)

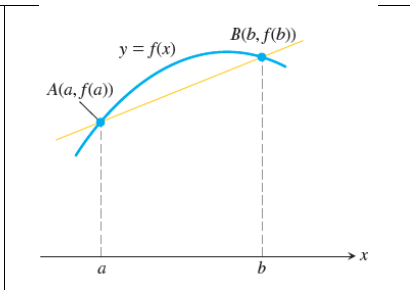

Figure (b)

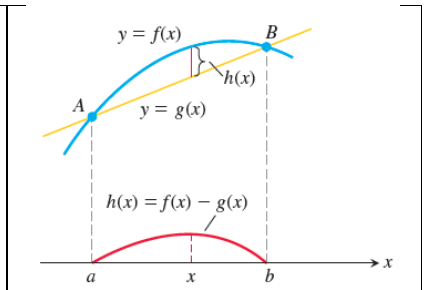

Figure (c)

Differentiating both sides of equation (2) with respect to x, we get

$$h'(x) = f'(x) - \frac{f(b)-f(a)}{b-a}$$

When $x = c$;

$$h'(c) = f'(c) - \frac{f(b)-f(a)}{b-a}$$

or, $\quad 0 = f'(c) - \frac{f(b)-f(a)}{b-a} \qquad\qquad [\because h'(c) = 0]$

$\therefore \quad f'(c) = \frac{f(b)-f(a)}{b-a}$

Physical Interpretation of LMV

Here $\frac{f(b)-f(a)}{b-a}$ represents average change in f over $[a, b]$ and $f'(c)$ represents an instantaneous change. Then the Mean Value Theorem says that at some interior point the instantaneous change must equal the average change over the entire interval.

Illustration

Very Lagrange's Mean Value Theorem for the function $f(x) = x^2$ in [1, 2].

Solution

i) Since the function $f(x) = x^2$ is a polynomial function, so it is continuous in [1, 2].
ii) Here $f'(x) = 2x$ which exists for all values of *x* in (1, 2).

By Lagrange's Mean Value Theorem, there exists a point *c* in $(1, 2)$ such that

$$f'(c) = \frac{f(2) - f(1)}{2 - 1}$$

i.e., $\quad 2c = \frac{2^2-1^2}{2-1}$

or, $\quad 2c = 3$

or, $\quad c = 3/2$ which lies in the interval (1, 2).

Thus, the Lagrange's Mean Value Theorem is verified.

Illustration

Very Lagrange's Mean Value Theorem for the function $f(x) = e^x$ on [0, 1].

Solution

i) Since the function $f(x) = e^x$ is continuous on [0, 1].
ii) Here $f'(x) = e^x$ which exists for all values of *x* in (0, 1).

By Lagrange's Mean Value Theorem, there exists a point *c* in $(0, 1)$ such that

$$f'(c) = \frac{f(1) - f(0)}{1 - 0}$$

i.e., $\quad e^c = \frac{e^1-e^0}{1-0}$

or, $\quad e^c = e - 1$

or, $e^c = 2.71828 - 1 = 1.71828$

or, $c \ln e = \ln 1.71828$

or, $c = 0.5413$ which lies in the interval (0, 1).

Thus, the Lagrange's Mean Value Theorem is verified.

Illustration

Verify Lagrange's Mean Value Theorem for the function $f(x) = x + \frac{1}{x}$ in $\left[-\frac{1}{2}, 2\right]$.

Solution

Since the function $f(x) = x + \frac{1}{x} = \frac{x^2+1}{x}$ is not continuous at $x = 0$ which is the point in the given interval $\left[-\frac{1}{2}, 2\right]$.

Thus, the condition for Lagrange's Mean Value Theorem is not satisfied for the given function in $\left[-\frac{1}{2}, 2\right]$.

Illustration

Very Lagrange's Mean Value Theorem for the following function in $[-1, 1]$.

$$f(x) = \begin{cases} x \sin 1/x & \text{for } x \neq 0 \\ 0 & \text{for } x = 0 \end{cases}$$

Solution

Since the given function is continuous at $x = 0$. Thus, the function is continuous in $[-1, 1]$.

But the given function is not differentiable at $x = 0$.

Thus, the condition for Lagrange's Mean Value Theorem is not satisfied.

Illustration

Show that $|\sin b - \sin a| \leq |b - a|$ by using Lagrange's Mean Value Theorem.

Solution

Let $f(x) = \sin x$ then $f'(x) = \cos x$.

Since the function $f(x) = \sin x$ is continuous in $[a, b]$ and differentiable in (a, b).

By Lagrange's Mean Value Theorem, there exists a point c in (a, b) such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

or, $\cos c = \frac{\sin b - \sin a}{b - a}$

or, $|\cos c| = \frac{|\sin b - \sin a|}{|b - a|}$

The value of cosine function lies between -1 and +1.

$\therefore \quad |\cos c| \leq 1$

i.e., $\quad \frac{|\sin b - \sin a|}{|b-a|} \leq 1$

$\Rightarrow |\sin b - \sin a| \leq |b - a|$

Cauchy's Mean Value Theorem

If $f(x)$ and $g(x)$ are two functions which are

i) continuous in $[a, b]$ and
ii) differentiable in (a, b)

then there exists at least one number c in (a, b) such that

$\frac{f(b)-f(a)}{g(b)-g(a)} = \frac{f'(c)}{g'(c)}$ where $g'(c) \neq 0$ in (a, b).

Proof: Let us define a function $F(x) = f(x)\{g(b) - g(a)\} - g(x)\{f(b) - f(a)\}$

This function is continuous in $[a, b]$ and differentiable in (a, b) since $f(x)$ and $g(x)$ are continuous in $[a, b]$ and differentiable in (a, b).

Here $F(a) = f(a)\{g(b) - g(a)\} - g(a)\{f(b) - f(a)\}$

$\quad = f(a)g(b) - g(a)f(b)$

and $F(b) = f(a)g(b) - g(a)f(b) = F(a)$

Then by Rolle's theorem, there exists c in (a, b) such that $F'(c) = 0$

Here, $\quad F'(x) = f'(x)\{g(b) - g(a)\} - g'(x)\{f(b) - f(a)\}$

Thus, $\quad f'(c)\{g(b) - g(a)\} - g'(c)\{f(b) - f(a)\} = 0$

or, $\quad f'(c)\{g(b) - g(a)\} = g'(c)\{f(b) - f(a)\}$

$\therefore \quad \frac{f'(c)}{g'(c)} = \frac{f(b)-f(a)}{g(b)-g(a)}$

Illustration

Verify Cauchy's Mean Value Theorem for the functions $f(x) = x^2$and $g(x) = 3x - 2$ in [1, 2].

Solution

i) Since $f(x)$ and $g(x)$ both are continuous in [1, 2].
ii) Here $f'(x) = 2x$ and $g'(x) = 3$ both exist in (1, 2).

By Cauchy's Mean Value Theorem, there exists a point c in $(1, 2)$ such that

$$\frac{f'(c)}{g'(c)} = \frac{f(2) - f(1)}{g(2) - g(1)}$$

i.e. $\quad \frac{2c}{3} = \frac{4-1}{4-1}$

$\therefore \qquad c = 3/2 \in (1, 2)$

Thus, the Cauchy's Mean Value Theorem is verified.

Illustration

Verify Cauchy's Mean Value Theorem for the functions $f(x) = x$ and $g(x) = x^2 - 2x$ in $[0, 2]$.

Solution

i) Since $f(x)$ and $g(x)$ both are continuous in [0, 2].

ii) Here $f'(x) = 1$ and $g'(x) = 2x - 2$ both exist in (0, 2).

However, $g'(x) = 0$ at $x = 1 \in (0, 2)$.

Thus, the Cauchy's Mean Value Theorem is not satisfied in [0, 2].

Exercise for Reader

1. Verify Lagrange's Mean Value Theorem for the function $f(x) = \ln x$ in $[1, e]$.
2. Prove that $\frac{b-a}{\sqrt{1-a^2}} < \sin^{-1} b - \sin^{-1} a < \frac{b-a}{\sqrt{1-b^2}}$ where $a < b < 1$. [Hint: assume $f(x) = \sin^{-1} x$ and apply Lagrange's Mean Value Theorem]
3. Verify Cauchy's Mean Value Theorem for the functions $f(x) = \sqrt{x}$ and $g(x) = 1/\sqrt{x}$ in $[a, b]$.