Lecture 1

Learning Objectives
At the end of this class, students should be able to:

- understand the concept of function
- understand the concept of limit
- solve the related problems

Functions
Let A and B be two nonempty sets. A function is a rule that assigns to each object in a set A exactly one object in a set B. The set A is called the domain of the function, and the set of assigned objects in B is called the range.

A function from A to B is denoted by $f: A \to B$.

Thus, we can say that a function is a correspondence between two sets of elements such that to each element x in the first set, there corresponds one and only one element $y = f(x)$ in the second set.

The following are few examples for correspondence.

- To each person, there corresponds a certain age.
- To each house in a city, there corresponds a plinth-area.
- To each student, there corresponds a grade-point average.

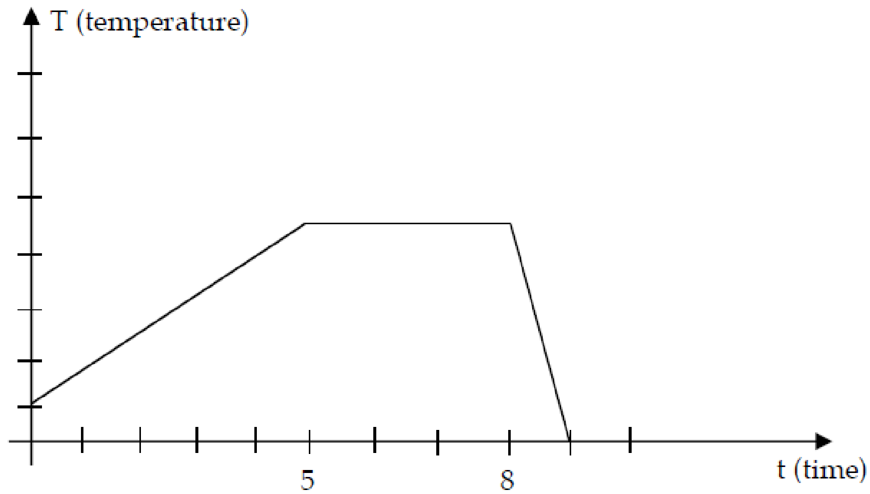
In the functional relation $y = f(x)$, the input values are domain values, and the output values are range values. This relation (equation) assigns each domain value x a range value y. The variable x is called an *independent variable* (since values can be independently assigned to x from the domain), and y is called a *dependent variable* (since the value of y depends on the value assigned to x). In general, any variable used as a placeholder for domain values is called an independent variable; any variable that is used as a placeholder for range values is called a dependent variable. For most functions that we will study in this course, the domain and range will be collections of real numbers.

Many formulas in mathematics can be expressed as a function, e.g., the area of a circle is a function of its radius. If we denote the radius of a circle by r then its area y can be written as $y = f(r) = \pi r^2$.

The Graph of a Function
The graph of a function $f(x)$ is the set of points $(x, f(x))$ in a coordinate plane, where x is in the domain of f.

Graphs are often used to describe the variation of physical quantities. For example, a scientist may use the following figure to indicate the temperature T of a certain solution at various time

t during an experiment. The sketch shows that the temperature increases gradually for time t = 0 to time t = 5, did not change between t = 5 and t = 8, and then decreased rapidly from t = 8 to t = 9.



Classification of Function

Functions are classified into two categories: Algebraic Functions and Transcendental Functions.

An **algebraic function** is a function that involves only algebraic operations, like, addition, subtraction, multiplication, and division, as well as fractional or rational exponents of independent variable x. The polynomial functions (e.g., $f(x) = 2x + 5,\ f(x) = x^2 + 3x + 2$), rational functions (e.g., $f(x) = \frac{2x+7}{x^2-1}$), and irrational functionals (e.g., $f(x) = \sqrt{x^2 - 9}$) fall under this category.

A **transcendental function** is a function that is not algebraic. The exponential functions (e.g., $f(x) = e^x,\ f(t) = 500(1.02)^t$), logarithm functions (e.g., $f(x) = \log(x + 3)$), trigonometric functions (e.g., $f(\theta) = \sin\theta$), and hyperbolic functions (e.g., $f(x) = \cosh x$) fall under this category.

Limit of a Function

The value of the function $f(x) = \dfrac{2x}{x+5}$ at x = 1 is $f(1) = \dfrac{2\times 1}{1+5} = \dfrac{2}{6} = \dfrac{1}{3}$. Here we can easily find the value of the function just by substituting x = 1. Sometimes, we can't work something out directly but we can see what it should be as we get closer and closer to that particular value from left-hand side and right-hand side. Let us try to understand the situation with the help of the following illustration.

Illustration

Determine the value of the function $f(x) = \dfrac{x^2 - 1}{x - 1}$ at x = 1.

Solution

Here $f(1) \quad = \dfrac{1^2 - 1}{1 - 1} \qquad = \dfrac{1 - 1}{1 - 1} \qquad = \dfrac{0}{0}$

We don't know the value of 0/0 (it is "indeterminate"), so we need another way of answering this. So instead of trying to work it out for x = 1, let's try approaching it closer and closer to 1 from both sides.

Let us prepare tables of values of x approaching 1 and corresponding values of $f(x) = \dfrac{x^2 - 1}{x - 1}$.

Approaching x = 1 from left

x	0.5	0.9	0.99	0.999	0.99990	0.99999
$f(x) = \dfrac{x^2 - 1}{x - 1}$	1.5	1.9	1.99	1.999	1.99990	1.99999

Approaching x = 1 from right

x	1.5	1.1	1.01	1.001	1.0001	1.00001
$f(x) = \dfrac{x^2 - 1}{x - 1}$	2.5	2.1	2.01	2.001	2.0001	2.00001

Now we see that as x gets close to 1, then $f(x) = \dfrac{x^2 - 1}{x - 1}$ gets close to 2.

We are now faced with an interesting situation:
When x = 1 we don't know the answer (it is indeterminate). But we can see that it is going to be 2. We want to give the answer "2" but can't, so instead mathematicians say exactly what is going on by using the special word "limit".

The limit of $f(x) = \dfrac{x^2 - 1}{x - 1}$ as x approaches 1 is 2.

It is written in symbols as: $\lim\limits_{x \to 2} \dfrac{x^2 - 1}{x - 1} = 2$

So, it is a special way of saying, "ignoring what happens when we get there, but as we get closer and closer the answer gets closer and closer to 2".

If a function f(x) approaches a limit L as x approaches c, we say that the limit of a function is L. Mathematically, it is written as

$$\lim_{x \to c} f(x) = L$$

Thus, for the limit of a function to exist as the independent variable approaches c, the left-hand and right-hand limits must be equal.

To state it more precisely:

$$\lim_{x \to c} f(x) = L$$

If and only if

$$\lim_{x \to c^-} f(x) = L \qquad \text{and} \qquad \lim_{x \to c^+} f(x) = L .$$

Illustration

Let $f(x) = \begin{cases} x^2 + 1 & \text{if } x < 2 \\ x - 1 & \text{if } x > 2 \end{cases}$. Find

a) $\lim_{x \to 2^-} f(x)$ b) $\lim_{x \to 2^+} f(x)$ c) $\lim_{x \to 2} f(x)$ d) $f(2)$

Solution

a) $\lim_{x \to 2^-} f(\text{x}) = \lim_{x \to 2} (\text{x}^2 + 1)$ $[\because f(\text{x}) = \text{x}^2 + 1 \text{ if x} < 2]$

$= (2)^2 + 1 = 5$

b) $\lim_{x \to 2^+} f(\text{x}) = \lim_{x \to 2} (\text{x} - 1)$ $[\because f(\text{x}) = \text{x} - 1 \text{ if x} > 2]$

$= 2 - 1 = 1$

c) Since the one-sided limits are not equal: $\lim_{x \to 2^-} f(\text{x}) \neq \lim_{x \to 2^+} f(\text{x})$, therefore, $\lim_{x \to 2} f(\text{x})$ does not exist.

d) Because the definition of f does not assign a value to f for x = 2 only for x < 2 and x > 2, $f(2)$ does not exist.

Limits that Do Not Exist

If the function $f(x)$ does not approach a definite number as *x* approaches *c*, we say that the limit of a function does not exist.

Illustration

Determine the value of the function $f(x) = \frac{1}{x^2}$ at x = 0.

Solution

Let us prepare tables of values of x approaching 0 and corresponding values of.

Approaching x = 0 from left

x	-0.1	-0.01	-0.001	-0.0001	-0.00001
$f(x) = \frac{1}{x^2}$	100	10,000	1,000,000	100,000,000	10,000,000,000

Approaching x = 0 from right

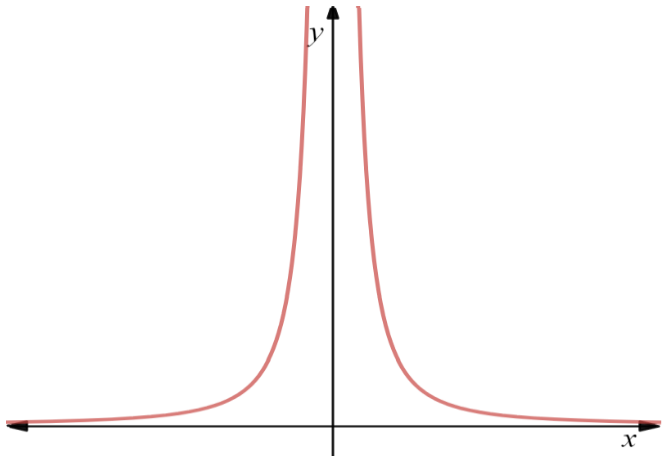
x	0.1	0.01	0.001	0.0001	0.00001
$f(x) = \frac{1}{x^2}$	100	10,000	1,000,000	100,000,000	10,000,000,000

Now we see that as x gets close to 0 from either side, $f(x) = \frac{1}{x^2}$ gets larger and larger, and does not approach to a fixed number.

Thus, $\lim_{x \to 0} \left(\frac{1}{x^2}\right) = \infty$

In such a case, we say that the limit does not exist.

The situation is described in the following graph



Approaching Infinity

Let us try to understand the concept with the help of the following calculation.

x	1	2	5	10	100	1000	10000	100000	1000000
1/x	1	0.5	0.2	0.1	0.01	0.001	0.0001	0.00001	0.000001

Here, we see that as x gets larger, 1/x tends towards 0.

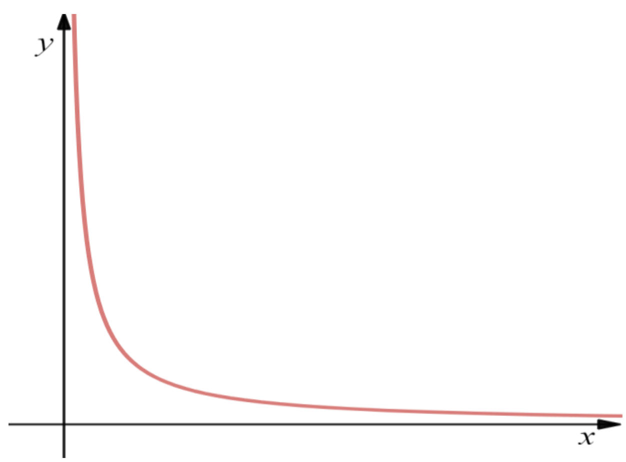
We can't say what happens when x gets to infinity but we can see that as x gets larger and larger, 1/x is going towards 0.

Thus, the limit of 1/x as x approaches infinity is 0. Mathematically,

$$\lim_{x \to \infty} \left(\frac{1}{x}\right) = 0$$

It is a mathematical way of saying "we are not talking about when x = ∞, but we know as x gets bigger, the answer gets closer and closer to 0".

The situation is described in the following graph



Lecture 2

Learning Objectives
At the end of this class, students should be able to:

- understand various techniques of finding limit
- solve the related problems

Techniques of Finding Limits

Let $f(x)$ be a given function of x. To evaluate $\lim_{x \to c} f(x)$, we use the following techniques:

1. a) Substitute c for x in $f(x)$ to find $f(c)$.
   b) If $f(c) = k$ (a finite number), then the required limit is f (c).
   c) If f (c) = $\frac{k}{0}$, the limit does not exist.

2. When $f(c)$ is not defined and takes the form: $\frac{0}{0}$.
   a) If possible, simplify $f(x)$ by factorization or rationalization so that we get the common factor $(x - c)$ in both numerator and denominator.
   b) As $(x - c) \neq 0$, so cancel out this common factor from numerator and denominator.
   c) Again find $f_1(c)$ where $f_1(x)$ is the expression which remains after canceling $(x - c)$ from numerator and denominator.
   d) If $f_1(c) = k$ (a finite number), then the required limit is $f_1(c)$.

3. If $\lim_{x \to \infty} f(x)$ is not defined and takes the form: $\frac{\infty}{\infty}$.
   Divide the numerator and denominator of $f(x)$ by the highest power of x that has appeared either in the numerator or in the denominator of $f(x)$, then apply the limit.

Note: $\frac{0}{0}$, $\frac{\infty}{\infty}$ *are known as indeterminate forms.*

Properties of Limits

Suppose $\lim_{x \to a} f(x) = L$ and $\lim_{x \to a} g(x) = M$ then

1. $\lim_{x \to a} [f(x)]^r = \left[\lim_{x \to a} f(x)\right]^r = L^r$ — r and a are real numbers

2. $\lim_{x \to a} cf(x) = c \lim_{x \to a} f(x) = cL$ — c and a are real numbers

3. $\lim_{x \to a} [f(x) \pm g(x)] = \lim_{x \to a} f(x) \pm \lim_{x \to a} g(x) = L \pm M$

4. $\lim_{x \to a} [f(x) g(x)] = \left[\lim_{x \to a} f(x)\right] \left[\lim_{x \to a} g(x)\right] = LM$

5. $\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)} = \frac{L}{M}$ — Provided $M \neq 0$

6. For every c in the in the trigonometric function's domain,

$$\lim_{x\to c}\sin x = \sin c \qquad \lim_{x\to c}\operatorname{cosec} x = \operatorname{cosec} c$$

$$\lim_{x\to c}\cos x = \cos c \qquad \lim_{x\to c}\sec x = \sec c$$

$$\lim_{x\to c}\tan x = \tan c \qquad \lim_{x\to c}\cot x = \cot c$$

Illustration

Evaluate $\lim_{x\to -2}\left(\frac{x^2+6x+8}{x+2}\right)$

Solution

Here $\lim_{x\to -2}\left(\frac{x^2+6x+8}{x+2}\right)$

The function $\frac{x^2+6x+8}{x+2}$ is not defined when x = -2, since $\frac{x^2+6x+8}{x+2} = \frac{0}{0}$ which is undefined.

Now, $\lim_{x\to -2}\left(\frac{x^2+6x+8}{x+2}\right) = \lim_{x\to -2}\frac{(x+2)(x+4)}{(x+2)}$

$= \lim_{x\to -2}(x+4)$

= -2 + 4 = 2

Illustration

Evaluate $\lim_{x\to\infty}\frac{x}{x^2+1}$

Solution

Here $\lim_{x\to\infty}\frac{x}{x^2+1} = \lim_{x\to\infty}\frac{\frac{x}{x^2}}{\frac{x^2}{x^2}+\frac{1}{x^2}}$

$= \lim_{x\to\infty}\frac{\frac{1}{x}}{1+\frac{1}{x^2}}$

$= \frac{\frac{1}{\infty}}{1+\frac{1}{\infty^2}}$

$= \frac{0}{1+0}$

$= 0$

Illustration

Evaluate $\lim\limits_{x \to 3} \frac{x-3}{\sqrt{x-2}-\sqrt{4-x}}$

Solution

Here, $\lim\limits_{x \to 3} \frac{x-3}{\sqrt{x-2}-\sqrt{4-x}}$

The function $\frac{x-3}{\sqrt{x-2}-\sqrt{4-x}}$ is not defined when x = 3, since $\frac{x-3}{\sqrt{x-2}-\sqrt{4-x}}$ = $\frac{0}{0}$ which is undefined.

Now, $\lim\limits_{x \to 3} \frac{x-3}{\sqrt{x-2}-\sqrt{4-x}}$ $= \lim\limits_{x \to 3} \frac{x-3}{\sqrt{x-2}-\sqrt{4-x}} \times \frac{\sqrt{x-2}+\sqrt{4-x}}{\sqrt{x-2}+\sqrt{4-x}}$

$$= \lim_{x \to 3} \frac{(x-3)(\sqrt{x-2}+\sqrt{4-x})}{(x-2)-(4-x)}$$

$$= \lim_{x \to 3} \frac{(x-3)(\sqrt{x-2}+\sqrt{4-x})}{2x-6}$$

$$= \lim_{x \to 3} \frac{(x-3)(\sqrt{x-2}+\sqrt{4-x})}{2(x-3)}$$

$$= \lim_{x \to 3} \frac{(\sqrt{x-2}+\sqrt{4-x})}{2}$$

$$= \frac{(\sqrt{3-2}+\sqrt{4-3})}{2}$$

= 2/2 = 1

Some Important Limits

1. $\lim\limits_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e$
2. $\lim\limits_{x\to 0}(1+x)^{1/x} = e$
3. $\lim\limits_{x\to 0}\frac{e^x-1}{x} = 1$
4. $\lim\limits_{x\to 0}\frac{\ln(1+x)}{x} = 1$
5. $\lim\limits_{x\to a}\frac{x^n-a^n}{x-a} = na^{n-1}$
6. $\lim\limits_{x\to 0}\frac{\sin x}{x} = 1$
7. $\lim\limits_{x\to 0}\frac{\cos x}{x} = 1$
8. $\lim\limits_{x\to 0}\frac{\tan x}{x} = 1$

Working Rule for Finding Left- and Right-Hand Limits
In the left-hand limit $\lim\limits_{x \to c^-} f(x) = L$, we can replace x by $c - h$ and make $h \to 0$.

That is, $\lim\limits_{x \to c^-} f(x) = \lim\limits_{h \to 0} f(c - h)$

Similarly, in the right-hand limit $\lim\limits_{x \to c^+} f(x) = L$, we can replace x by $c + h$ and make $h \to 0$.

That is, $\lim\limits_{x \to c^+} f(x) = \lim\limits_{h \to 0} f(c + h)$

Illustration
Does $\lim\limits_{x \to 0} \frac{e^{1/x}-1}{e^{1/x}+1}$ exist?
Solution
Here, $f(x) = \frac{e^{1/x}-1}{e^{1/x}+1}$
Left-Hand Limit at $x = 0$ is

$$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} \left(\frac{e^{1/x} - 1}{e^{1/x} + 1}\right)$$

Putting, $x = 0 - h$ then $h \to 0$ when $x \to 0^-$, we get

$$\lim_{x \to 0^-} f(x) = \lim_{h \to 0} \left(\frac{e^{-1/h} - 1}{e^{-1/h} + 1}\right)$$

$$= \frac{e^{-1/0}-1}{e^{-1/0}+1}$$

$$= \frac{e^{-\infty}-1}{e^{-\infty}+1}$$

$$= \frac{0-1}{0+1}$$

$$= -1$$

Right-Hand Limit at $x = 0$ is

$$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} \left(\frac{e^{1/x} - 1}{e^{1/x} + 1}\right)$$

Putting, $x = 0 + h$ then $h \to 0$ when $x \to 0^+$, we get

$$\lim_{x \to 0^+} f(x) = \lim_{h \to 0} \left(\frac{e^{1/h} - 1}{e^{1/h} + 1}\right)$$

$$= \lim_{h\to 0} \frac{e^{1/h}(1-\frac{1}{e^{1/h}})}{e^{1/h}(1+\frac{1}{e^{1/h}})}$$

$$= \lim_{h\to 0} \frac{1-\frac{1}{e^{1/h}}}{1+\frac{1}{e^{1/h}}}$$

$$= \frac{1-\frac{1}{e^{\infty}}}{1+\frac{1}{e^{\infty}}}$$

$$= \frac{1-\frac{1}{\infty}}{1+\frac{1}{\infty}}$$

$$= \frac{1-0}{1+0}$$

$$= 1$$

Since $\lim_{x\to 0^-} f(x) \neq \lim_{x\to 0^+} f(x)$

So, the limit does not exist.

The Sandwich Theorem: Suppose $g(x) \leq f(x) \leq h(x)$ for all x in some open interval containing c, except possibly at $x = c$ itself. Suppose $\lim_{x\to c} g(x) = \lim_{x\to c} h(x) = L$ then $\lim_{x\to c} f(x) = L$.

Illustration

Evaluate $\lim_{x\to 0} x\sin(1/x)$.

Solution

Let $g(x) = -x, f(x) = x\sin\left(\frac{1}{x}\right)$, and $h(x) = x$. Plotting the graph, we get the following figure:

From the graph,

$\lim_{x\to 0} g(x) = 0$ and $\lim_{x\to 0} f(x) = 0$

Thus, by sandwich theorem, $\lim_{x\to 0} f(x) = \lim_{x\to 0} x\sin(1/x) = 0$

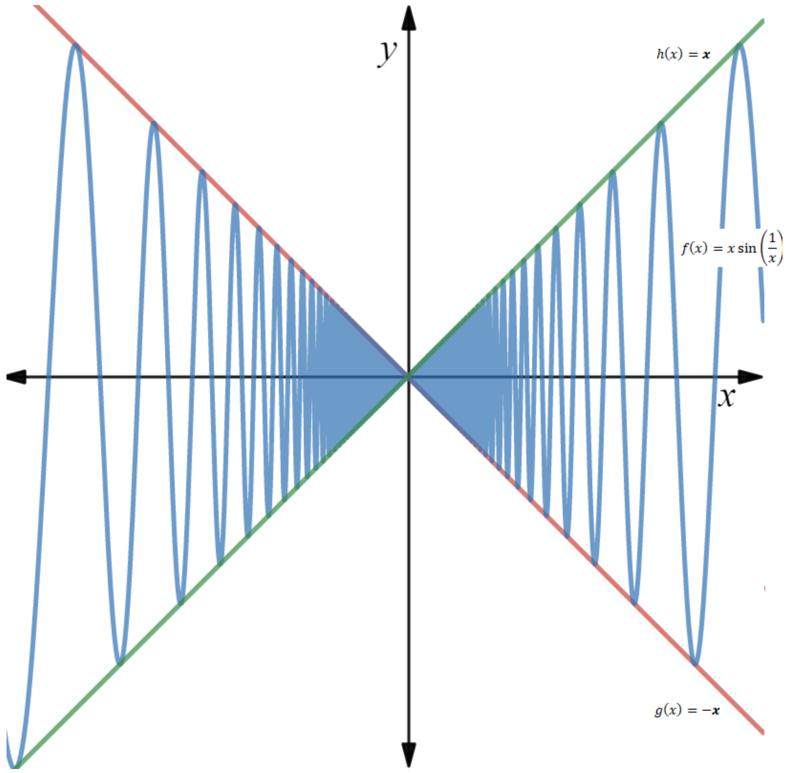
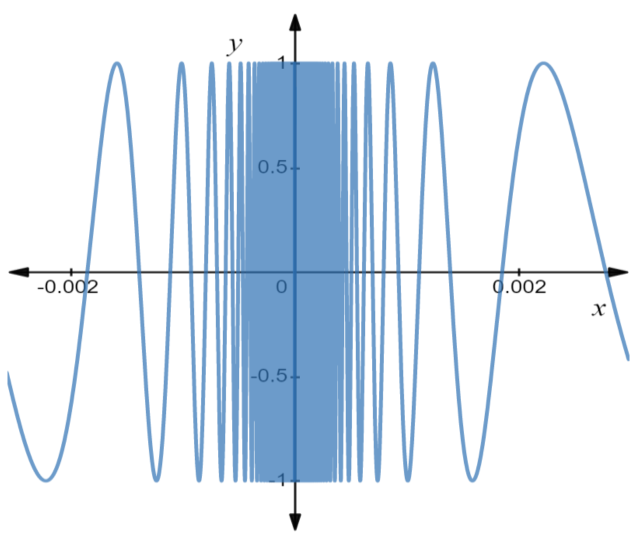


Illustration
Evaluate $\lim_{x \to 0} \sin(1/x)$
Solution
The graph of the given function $f(x) = \sin(1/x)$ is shown in the following figure:



As x gets close to zero, its reciprocal $1/x$ grows without bound and the curve (function) oscillates up and down between -1 and +1, the rapidity of the oscillations becoming greater and greater as x

gets closer to 0. The function can't attain a single number L as *x* approaches zero from either side. Thus, the function has neither left-hand limit nor a right-hand limit.

Exercise for Reader

1. Let $f(x)=\begin{cases}1-x^2 & \text{if } x\le 0\\ 1+x^2 & \text{if } x>0\end{cases}$. Find

   a) $\lim_{x\to 0^-} f(x)$ b) $\lim_{x\to 0^+} f(x)$ c) $\lim_{x\to 0} f(x)$ d) f (0)

2. Let $f(x)=\begin{cases}-2x+4 & \text{if } x<1\\ 4 & \text{if } x=1\\ x^2+1 & \text{if } x>1\end{cases}$. Evaluate $\lim_{x\to 1} f(x)$, if it exists.

3. Does $\lim_{x\to 3}\frac{|x-3|}{(x-3)}$ exist?

4. Evaluate $\lim_{x\to 0} x^2\sin(1/x)$

Lecture 3

Learning Objectives
At the end of this class, students should be able to:

- understand the concept of continuity
- identify the points of discontinuity
- identify the interval on which the given function is continuous
- solve the related problems

Continuity

A function f is continuous at $x = c$ if

i) $\lim_{x \to c} f(x)$ exists

ii) $f(c)$ is defined, and

iii) $\lim_{x \to c} f(x) = f(c)$.

Thus, a function $f(x)$ is continuous at $x = c$ if

$$\lim_{x \to c^-} f(x) = \lim_{x \to c^+} f(x) = f(c).$$

Equivalently, we say $f(x)$ is continuous at $x = c$ if for $h > 0$,

$$\lim_{h \to 0} f(c-h) = \lim_{h \to 0} f(c+h) = f(c).$$

Illustration
Check the continuity of the following function at $x = 1$.

$$f(x) = \begin{cases} 2x - 1 & \text{if } 0 \le x \le 1 \\ x & \text{if } x > 1 \end{cases}$$

Solution
Here, left hand limit at $x = 1$ is

$$\lim_{x \to 1^-} f(x) = \lim_{x \to 1} (2x-1)$$
$$= 2 - 1 = 1$$

Right hand limit at $x = 1$ is

$$\lim_{x \to 1^+} f(x) = \lim_{x \to 1} (x)$$
$$= 1$$

and $f(1) = 1$
Since $\lim_{x \to 1^-} f(x) = \lim_{x \to 1^+} f(x) = f(1)$, therefore, the given function is continuous at $x = 1$.

Illustration
Check the continuity of the following function at x = 0.

$$f(x) = \begin{cases} \frac{x-6}{x-3} & \text{for } x < 0 \\ 2 \text{ for } x = 0 \\ \sqrt{4 + x^2} \text{ for } x > 0 \end{cases}$$

Solution
Here, left hand limit is

$$\lim_{x \to 0^-} f(x) = \lim_{x \to 0} \frac{x-6}{x-3} = \frac{0-6}{0-3} = 2$$

Right hand limit is

$$\lim_{x \to 0^+} f(x) = \lim_{x \to 0} \sqrt{4+x^2} = \sqrt{4+0^2} = 2$$

The value of the function at x = 0 is *f*(0) = 2

Since $\lim_{x \to 0^-} f(x) = \lim_{x \to 0^+} f(x) = f(0)$ therefore the given function is continuous at x = 0.

Illustration
Check the continuity of the following function at x = 0.

$$f(x) = \begin{cases} x \sin 1/x & \text{for } x \neq 0 \\ 0 & \text{for } x = 0 \end{cases}$$

Solution
Here,

$$\lim_{x \to 0} x \sin(1/x) = 0$$

[By sandwich theorem]

$$f(0) = 0$$

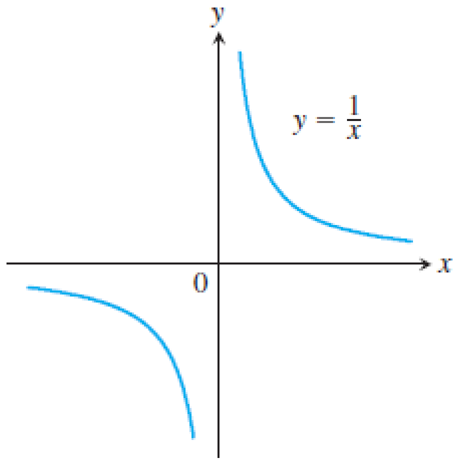
Thus, the given function is continuous at x = 0.

Properties of Continuous Functions
If the functions *f* and *g* are continuous at $x = c$, then the following algebraic combinations are continuous at $x = c$.

1.	Sums	$f + g$
2.	Differences	$f - g$
3.	Constant multiples	$k.f$, for any number k
4.	Products	$f.g$
5.	Quotients	f/g, provided $g(c) \neq 0$
6.	Powers	f^n, n is a positive integer
7.	Roots	$\sqrt[n]{f}$, provided it is defined on an open interval containing c, where n is a positive integer

Continuous Functions
A function f is continuous on its domain if it is continuous at every point of the domain. If a function is discontinuous at one or more points of its domain, we say it is a discontinuous function. Polynomial functions are continuous in the entire domain, since these functions are defined for every real number and $\lim_{x \to a} f(x) = f(a)$ for all real a.

From the following figure, we see that the function $f(x) = 1/x$ is a continuous function because it is continuous at every point of its domain. It has a point of discontinuity at $x = 0$, however, because it is not defined there; that is, it is discontinuous on any interval containing $x = 0$.



Illustration

Find the discontinuities of the function $f(x) = \dfrac{x-2}{x^2 - 2x - 35}$.

Solution

The given function is not defined if the denominator becomes zero.

i.e., $x^2 - 2x - 35 = 0$

or, $(x-7)(x+5) = 0$

Thus, the denominator becomes zero for $x = 7$ or $x = -5$. Since the given function is not defined when $x = 7$ or $x = -5$, the function is discontinuous at these two points.

Illustration

Find the intervals on which the function $f(x) = \frac{x}{9x^2 - 3x}$ is continuous.

Solution

The given function is discontinuous if the denominator becomes zero,

i.e., $9x^2 - 3x = 0$

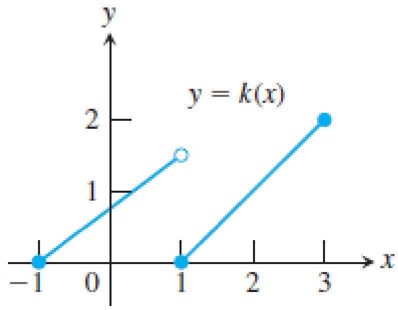
or, $3x(3x - 1) = 0$

Thus, the denominator becomes zero when $x = 0$ or $x = 1/3$. This means the given function is continuous except at $x = 0$ and $x = 1/3$.

Hence, the given function is continuous on the intervals $(-\infty, 0)$, $(0, 1/3)$, and $(1/3, \infty)$.

Exercise for Reader

1. whether the function graphed in the following figure is continuous on the interval [-1, 3]. If not, where does it fail to be continuous and why?



2. Check the continuity of the function at x = 2.

$$f(x)=\begin{cases}\dfrac{x^2-4}{x-2} & \text{for } x\neq 2\\ 3 & \text{for } x=2\end{cases}$$

3. Check the continuity of the function at x = 3.

$$f(x)=\begin{cases}3x & \text{for } x<3\\ 9 & \text{for } x=3\\ 2x+3 & \text{for } x>3\end{cases}$$

4. In the following exercises, determine whether there are any discontinuities and, if so, where they occur?

   a) $f(x)=\ x^2-5x+7$ b) $f(x)=\dfrac{5x+3}{x^3-x}$

5. In the following exercises, find the intervals on which the function is continuous.

   a) $f(x)=\dfrac{5x}{x+2}$ b) $f(x)=\dfrac{3x+2}{x^2-5x}$

Lecture 4

Learning Objectives
At the end of this class, students should be able to:

- understand the concept of left-hand and right-hand derivatives
- familiar with various differentiation rules
- solve the related problems

Introduction
Let us start with the following question.
The population of a city is estimated by the function $P = f(t) = 1.2e^{0.045t}$ where P equals the population (in millions) and t equals time measured in years since 2010.
a) Determine the expression for the rate of change in the population.
b) At what rate is the population expected to be changing in 2020.

To answer the above problem, we must have the knowledge of the derivative. If $y = f(x)$ is a given function then the symbol $\frac{dy}{dx}$ is used to express the derivative of *y* with respect to *x*. The interpretation $\frac{dy}{dx}$ is the change in the dependent variable *y* due to small unit change in independent variable *x*. This type of change is also known as *instantaneous rate of change* or just *rate of change*.

The process of obtaining the derivative $f'(x)$ is called *differentiation* of the function *f* (*x*). The other symbols that are used for derivative are $\frac{df(x)}{dx}$, or *y*', or y_1, or simply by *f* '.

Derivative at a Point
The derivative of a function $f(x)$ at a point $x = a$, denoted by $f'(a)$, defined as

$$f'(a) = \lim_{h\to 0}\frac{f(a+h)-f(a)}{h}; \qquad h > 0$$

A function $f(x)$ is said to be differentiable at a point $x = a$ if for $h > 0$,

$$\lim_{h\to 0}\frac{f(a-h)-f(a)}{-h} = \lim_{h\to 0}\frac{f(a+h)-f(a)}{h}$$

The left-hand side is known as left-hand derivative (LHD) and the right-hand side is known as right-hand derivative (RHD) of a function $f(x)$ at a point $x = a$.

Whenever LHD ≠ RHD, we say that $f(x)$ is not differentiable at a point $x = a$.

Relation between Continuity and Differentiability
If a function $f(x)$ is differentiable at any point of its domain, it is necessarily continuous at that point; but the converse may not be true.

Proof: Suppose the function $f(x)$ be differentiable at any point $x = a$ of its domain. Then

$$f'(a) = \lim_{h\to 0}\frac{f(a+h)-f(a)}{h}$$

Now, $\lim_{h \to 0}[f(a+h) - f(a)] = \lim_{h \to 0}\left[\frac{f(a+h)-f(a)}{h} \times h\right]$

$$= \lim_{h \to 0}\left[\frac{f(a+h)-f(a)}{h}\right] \times \lim_{h \to 0}(h)$$

$$= f'(a) \times 0 \ = 0$$

Thus, $\lim_{h \to 0}[f(a+h) - f(a)] = 0$

or, $\lim_{h \to 0} f(a+h) = f(a)$

or, $\lim_{x \to a} f(x) = f(a)$ $\quad [\because a + h = x]$

Hence, $f(x)$ is continuous at $x = a$ whenever $f'(a)$ exists.

The converse of this theorem may not be true. We prove this with the help of following example.

Consider the function

$$f(x) = |x| = \begin{cases} -x, & x < 0 \\ x, & x \geq 0 \end{cases}$$

For continuity at $x = 0$,

LHL = $\lim_{x \to 0^-} f(x) = \lim_{h \to 0} f(0 - h)$

$$= \lim_{h \to 0} f(-h) = \lim_{h \to 0} \{-(-h)\} = 0$$

RHL = $\lim_{x \to 0^+} f(x) = \lim_{h \to 0} f(0 + h)$

$$= \lim_{h \to 0} f(h) = \lim_{h \to 0} (h) = 0$$

and $f(0) = 0$

Therefore, the function $f(x) = |x|$ is continuous at $x = 0$.

For differentiability at $x = 0$,

LHD = $\lim_{h \to 0} \frac{f(0-h)-f(0)}{-h} = \lim_{h \to 0} \frac{f(-h)-f(0)}{-h}$

$$= \lim_{h \to 0} \frac{\{-(-h)\}-0}{-h} = -1, \text{ and}$$

RHD = $\lim_{h \to 0} \frac{f(0+h)-f(0)}{h} = \lim_{h \to 0} \frac{f(h)-f(0)}{h}$

$$= \lim_{h \to 0} \frac{h-0}{h} = 1$$

Since at $x = 0$, LHD ≠ RHD. Thus, the function $f(x) = |x|$ is not differentiable at $x = 0$ although it is continuous at $x = 0$.

Illustration
Examine the continuity and differentiability of the following function at the origin.

$$f(x) = \begin{cases} x^{1/3}, & x < 0 \\ x^{2/3}, & x \geq 0 \end{cases}$$

Solution
For continuity at $x = 0$,

LHL = $\lim\limits_{x \to 0^-} f(x) = \lim\limits_{x \to 0} (x^{1/3})$

$= 0$

RHL = $\lim\limits_{x \to 0^+} f(x) = \lim\limits_{x \to 0} (x^{2/3})$

$= 0$

and $f(0) = 0$

Therefore, the function $f(x)$is continuous at $x = 0$.

For differentiability at $x = 0$,

LHD = $\lim\limits_{h \to 0} \frac{f(0-h)-f(0)}{-h} = \lim\limits_{h \to 0} \frac{f(-h)-f(0)}{-h}$

$$= \lim_{h \to 0} \frac{(-h^{1/3})-0}{-h} = \lim_{h \to 0} \frac{1}{h^{2/3}}$$

$$= \frac{1}{0} = \infty$$

Since at $x = 0$, LHD does not exist, so the given function is not differentiable at $x = 0$ although it is continuous at $x = 0$.

Illustration
Examine the continuity and differentiability of the following function at $x = \pi/2$.

$$f(x) = \begin{cases} 1 + \sin x, & 0 \leq x < \pi/2 \\ 2 + (x - \pi/2)^2, & \frac{\pi}{2} \leq x < \infty \end{cases}$$

Solution
For continuity at $x = \pi/2$,

LHL = $\lim\limits_{x \to \pi/2^-} f(x) = \lim\limits_{x \to \pi/2} (1 + \sin x)$

$= 1 + 1 = 2$

RHL = $\lim\limits_{x \to \pi/2^+} f(x) = \lim\limits_{x \to \pi/2} \{2 + (x - \pi/2)^2\}$

$= 2 + 0 = 2$

and $f(\pi/2) = 2 + (\pi/2 - \pi/2)^2 = 2$

Therefore, the function $f(x)$is continuous at $x = \pi/2$.

For differentiability at $x = \pi/2$,

LHD = $\lim\limits_{h \to 0} \frac{f(\pi/2 - h) - f(\pi/2)}{-h} = \lim\limits_{h \to 0} \frac{1 + \sin(\pi/2 - h) - 2}{-h}$

$= \lim\limits_{h \to 0} \frac{\cos h - 1}{-h} = \lim\limits_{h \to 0} \frac{1 - 2\sin^2(\frac{h}{2}) - 1}{-h}$

$= 2 \lim\limits_{h \to 0} \left[\frac{\sin h/2}{h/2}\right]^2 \times \left(\frac{h}{2}\right)$ $\left[\because \lim\limits_{h \to 0} \frac{\sin h/2}{h/2} = 1\right]$

$= 2 \times 1 \times 0 = 0$

RHD = $\lim\limits_{h \to 0} \frac{f\left(\frac{\pi}{2} + h\right) - f(\pi/2)}{h} = \lim\limits_{h \to 0} \frac{\{2 + (\pi/2 + h - \pi/2)^2\} - 2}{h}$

$= \lim\limits_{h \to 0} \frac{2 + h^2 - 2}{h} = \lim\limits_{h \to 0} (h)$

$= 0$

Since LHD = RHD, so the given function is differentiable at $x = \pi/2$.

Differentiation Rules

We use various rules for finding derivatives.

1. Derivative of a Constant Function: Derivative of a constant function is always zero. If $f(x) = c$, where c is any constant, then $f'(x) = 0$.
2. Power Rule: If $f(x) = x^n$, where n is a real number, then $f'(x) = nx^{n-1}$.
3. Sum or Difference Rule: If $f(x) = u(x) \pm v(x)$, where u and v are differentiable functions, then $f'(x) = u'(x) \pm v'(x)$
4. Product Rule: If $f(x) = u(x).v(x)$, where u and v are differentiable functions, then

   $f'(x) = u(x)v'(x) + v(x)u'(x)$.

5. Quotient Rule: If $f(x) = u(x)/v(x)$, where u and v are differentiable functions and $v(x) \neq 0$, then $f'(x) = \frac{v(x)u'(x) - u(x)v'(x)}{[v(x)]^2}$.
6. General Power Rule: If $f(x) = [u(x)]^n$, where u is a function of x and n is a real number, then $f'(x) = n[u(x)]^{n-1}.u'(x)$
7. Chain Rule: If $z = f(y)$, and $y = g(x)$, then the derivative of z with respect to x is given by the formula: $\frac{dz}{dx} = \frac{dz}{dy}.\frac{dy}{dx}$
8. Derivative of Exponential Function: Let $f(x) = e^x$ then $f'(x) = e^x$ and if $f(x) = e^{u(x)}$ then $f'(x) = e^{u(x)} \times \frac{d}{dx}(u(x))$, where u is a function of x.

   Note: If $f(x) = a^x$ then $f'(x) = a^x \ln a$
9. Derivative of Logarithmic Function: Let $f(x) = \ln x$ then $f'(x) = \frac{1}{x}, x > 0$ and similarly if $f(x) = \ln u(x)$ then $f'(x) = \frac{1}{u(x)}\left[\frac{d}{dx}\{u(x)\}\right]$, where u is a function of x.

   Note: If $f(x) = \log_a x$ then $f'(x) = \frac{1}{x}(\log_a e), x > 0$
10. Derivative of Trigonometric Functions: We use the following formulas for finding the derivative of trigonometric functions.

    a) $\frac{d}{dx}(sinx) = cosx$

    b) $\frac{d}{dx}(cosx) = -sinx$

    c) $\frac{d}{dx}(tanx) = sec^2x$

    d) $\frac{d}{dx}(cosecx) = -cosecx.cotx$

    e) $\frac{d}{dx}(secx) = secx.tanx$

    f) $\frac{d}{dx}(cotx) = -cosec^2x$

11. Derivative of Inverse Circular Functions: We use the following formulas for finding the

derivative of Inverse Circular functions.

a) $\frac{d}{dx}(sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}$, $(|x| < 1)$

b) $\frac{d}{dx}(cos^{-1} x) = \frac{-1}{\sqrt{1-x^2}}$, $(|x| < 1)$

c) $\frac{d}{dx}(tan^{-1} x) = \frac{1}{1+x^2}$

d) $\frac{d}{dx}(cosec^{-1} x) = \frac{-1}{x\sqrt{x^2-1}}$, $(x > 1)$

e) $\frac{d}{dx}(sec^{-1} x) = \frac{1}{x\sqrt{x^2-1}}$, $(x > 1)$

f) $\frac{d}{dx}(cot^{-1} x) = \frac{-1}{1+x^2}$

Exercise for Reader

1. Examine the continuity and differentiability of the following function at $x = 0$.

$$f(x) = \begin{cases} x \sin\left(\frac{1}{x}\right) & \text{for } x \neq 0 \\ 0 & \text{for } x = 0 \end{cases}$$

2. Examine the continuity and differentiability of the following function at $x = 1$ and at $= 2$.

$$f(x) = \begin{cases} 5x - 4 & \text{for } 0 < x \leq 1 \\ 4x^2 - 3x & \text{for } 1 \leq x \leq 2 \\ 3x + 4 & \text{for } x \geq 2 \end{cases}$$

3. Find the derivative of the following functions with respect to x.
   i) $f(x) = 5x^{3/2} - 7x + 8$
   ii) $f(x) = (4x^2 - 3x)(x^3 - 8x^2 + 12)$
   iii) $f(x) = \frac{5x^2-7x}{3x^2+2}$
   iv) $f(x) = (3 - 2x)^5$
   v) $f(x) = \frac{7x^2}{4e^x-x}$
   vi) $f(x) = \frac{1+ln\,x}{x^2-ln\,x}$
   vii) $y = \sqrt{x}.cos(2x^2 + 5)$
   viii) $y = tan^{-1}\frac{3x+2}{4}$
   ix) $y = cos(ln\,sec\,x)$

4. Find the value of $\frac{dy}{dx}$ if
   i) $y = x^2y^3 + x^3y^2$
   ii) $e^{xy} = xy^3$
   iii) $y = \frac{(ln\,x)^x}{2^{3x+1}}$
   iv) $x + y = \cos(x - y)$

Lecture 5

Learning Objectives
At the end of this class, students should be able to:

- find higher order derivatives
- use Leibnitz's theorem
- solve the related problems

Higher Order Derivatives

Let $y = f(x)$ be a differentiable function. Then $y' = \frac{d}{dx}\{f(x)\}$ is called the first order derivative of $y = f(x)$. If we differentiate it again, we get $y'' = \frac{d}{dx}(y')$ which is called the second order derivative of $y = f(x)$. Similarly, $y''' = \frac{d}{dx}(y'')$ is the called the third order derivative of $y = f(x)$. In general, the notation for the n[th] order derivative is $y^{(n)}$ or y_n or $f^n(x)$ or $\frac{d^ny}{dx^n}$ or D^ny.

Determination n[th] Derivative: Some Standard Results

1. Determination of the n[th] Derivative of $(\boldsymbol{ax+b})^{\boldsymbol{m}}$

Let $y = (ax+b)^m$ then

$$y_1 = ma(ax+b)^{m-1};$$

$$y_2 = m(m-1)a^2(ax+b)^{m-2};\ y_3 = m(m-1)(m-2)a^3(ax+b)^{m-3} \text{ and so on}$$

In general, $y_n = m(m-1)(m-2)\ .\ .\ .\ (m-n+1)a^n(ax+b)^{m-n}$

i) In case, m is a positive integer greater than n, y_n can be written as

$$y_n = \frac{m(m-1)(m-2)\ .\ .\ .\ (m-n+1)(m-n)(m-n-1)\ .\ .\ .\ 3.2.1\ a^n(ax+b)^{m-n}}{(m-n)(m-n-1)\ .\ .\ .\ 3.2.1}$$

$$y_n = \frac{m!\,a^n(ax+b)^{m-n}}{(m-n)!}$$

ii) In case, m = n, y_n can be written as

$$y_n = n!\,a^n$$

iii) In case, $m = -1$ (i.e., $y = \frac{1}{ax+b}$), y_n can be written as

$$y_n = (-1)^n 1.2.3\ .\ .\ .\ n\,a^n(ax+b)^{-1-n}$$

$$y_n = \frac{(-1)^n\,a^n n!}{(ax+b)^{n+1}}$$

iv) In case, $a = 1$ and $b = 0$ (i.e., $y = x^m$), y_n can be written as

$$y_n = m(m-1)(m-2)\ .\ .\ .\ (m-n+1)\quad (x)^{m-n}$$

2. Determination of the n^th^ Derivative of $\ln(ax+b)$

Let $y = \ln(ax+b)$ then

$y_1 = \frac{a}{ax+b} = a(ax+b)^{-1}$;

$y_2 = (-1)a^2(ax+b)^{-2}$; $y_3 = (-1)^2 1.2\, a^3(ax+b)^{-3}$; $y_4 = (-1)^3 1.2.3\, a^4(ax+b)^{-4}$ and so on

In general, $y_n = (-1)^{n-1}1.2.3\,.\,.\,.\,(n-1)\,a^n(ax+b)^{-n}$

$$y_n = (-1)^{n-1}\frac{(n-1)!\,a^n}{(ax+b)^n}$$

In case, $a = 1$ and $b = 0$ (i.e., $y = \ln x$), y_n can be written as

$$y_n = (-1)^{n-1}\frac{(n-1)!}{x^n}$$

3. Determination of the n^th^ Derivative of a^{mx}

Let $y = a^{mx}$ then

$y_1 = ma^{mx}\ln a$;

$y_2 = m^2a^{mx}(\ln a)^2$; $y_3 = m^3a^{mx}(\ln a)^3$ and so on

In general, $y_n = m^na^{mx}(\ln a)^n$

i) In case, $m = 1$ (i.e., $y = a^x$), y_n can be written as

$$y_n = a^x(\ln a)^n$$

ii) In case, $a = e$ (i.e., $y = e^{mx}$), y_n can be written as

$$y_n = m^ne^{mx}$$

iii) In case, $a = e$ and $m = 1$ (i.e., $y = e^x$), y_n can be written as

$$y_n = e^x$$

4. Determination of the n^th^ Derivative of $\sin(ax+b)$

Let $y = \sin(ax+b)$ then

$y_1 = a\cos(ax+b) = a\sin\left[\frac{\pi}{2} + (ax+b)\right]$;

$y_2 = a^2\cos\left[\frac{\pi}{2} + (ax+b)\right] = a^2\sin\left[\frac{2\pi}{2} + (ax+b)\right]$;

$y_3 = a^3\cos\left[\frac{2\pi}{2} + (ax+b)\right] = a^3\sin\left[\frac{3\pi}{2} + (ax+b)\right]$ and so on

In general, $y_n = a^n\sin\left[\frac{n\pi}{2} + (ax+b)\right]$

In case, $a = 1$ and $b = 0$ (i.e., $y = \sin x$), y_n can be written as

$$y_n = \sin\left[\frac{n\pi}{2} + x\right]$$

5. Determination of the n^th^ Derivative of $e^{ax}\sin(bx+c)$

Let $y = e^{ax}\sin(bx+c)$ then

$y_1 = ae^{ax}\sin(bx+c) + be^{ax}\cos(bx+c) = e^{ax}[a\sin(bx+c) + b\cos(bx+c)]$;

Putting $a = r\cos\theta$ and $b = r\sin\theta$, then $r = \sqrt{a^2+b^2}$ and $\theta = \tan^{-1}(b/a)$. Thus,

$y_1 = e^{ax}[r\cos\theta\sin(bx+c) + r\sin\theta\cos(bx+c)] = re^{ax}\sin(bx+c+\theta)$;

$y_2 = r[ae^{ax}\sin(bx+c+\theta) + b\,e^{ax}\cos(bx+c+\theta)] = r^2e^{ax}\sin(bx+c+2\theta)$;

$y_3 = r^3e^{ax}\sin(bx+c+3\theta)$ and so on

In general, $y_n = r^n e^{ax}\sin(bx+c+n\theta)$

Where $r = \sqrt{a^2+b^2}$ and $\theta = \tan^{-1}(b/a)$

6. Determination of the n^th^ Derivative of $\frac{1}{x^2+a^2}$

Let $y = \frac{1}{x^2+a^2} = \frac{1}{(x-ia)(x+ia)}$ where $i^2 = -1$

or, $y = \frac{1}{2ai}\left[\frac{1}{x-ia} - \frac{1}{x+ia}\right]$, then

$$y_n = \frac{(-1)^n n!}{2ai}\left[\frac{1}{(x-ia)^{n+1}} - \frac{1}{(x+ia)^{n+1}}\right]$$

or, $y_n = \frac{(-1)^n n!}{2ai}\left[(x-ia)^{-(n+1)} - (x+ia)^{-(n+1)}\right]$ (i)

Putting $x = r\cos\theta$ and $a = r\sin\theta$, then $r = \sqrt{x^2+a^2}$ and $\theta = \tan^{-1}(a/x)$. Thus,

$$(x-ia)^{-(n+1)} = r^{-(n+1)}[\cos\theta - i\sin\theta]^{-(n+1)}$$
$$= r^{-(n+1)}[\cos(n+1)\theta + i\sin(n+1)\theta]$$

[using De Moivre's Theorem]

$$(x+ia)^{-(n+1)} = r^{-(n+1)}[\cos\theta + i\sin\theta]^{-(n+1)}$$
$$= r^{-(n+1)}[\cos(n+1)\theta - i\sin(n+1)\theta]$$

Then from equation (i), we get

$$y_n = \frac{(-1)^n n!}{2ai}\left[r^{-(n+1)}2i\sin(n+1)\theta\right]$$

$$y_n = \frac{(-1)^n n!}{a}\, r^{-(n+1)}\sin(n+1)\theta$$

$$y_n = \frac{(-1)^n n!}{a}\left(\frac{a}{\sin\theta}\right)^{-(n+1)}\sin(n+1)\theta$$

$$y_n = \frac{(-1)^n n!}{a^{n+2}}\sin^{n+1}\theta\sin(n+1)\theta$$

Where $r = \sqrt{x^2+a^2}$ and $\theta = \tan^{-1}(a/x)$

Illustration

Find the n^th^ derivative of $\frac{x^n}{x-1}$.

Solution

Let $y = \frac{x^n}{x-1}$

or, $y = (x^{n-1} + x^{n-2} + x^{n-3} + \ . \ . \ . +1) + \frac{1}{x-1}$. Then

$y_n = D^n(x^{n-1} + x^{n-2} + x^{n-3} + \ . \ . \ . +1) + D^n\left(\frac{1}{x-1}\right)$

or, $y_n = 0 + \frac{(-1)^n \, n!}{(x-1)^{n+1}}$ $\left[\because \ D^n\left(\frac{1}{ax+b}\right) = \frac{(-1)^n \, a^n n!}{(ax+b)^{n+1}}\right]$

or, $y_n = \frac{(-1)^n \, n!}{(x-1)^{n+1}}$

Illustration

Find the n^th^ derivative of $\frac{1}{1-5x+6x^2}$.

Solution

Let $y = \frac{1}{1-5x+6x^2} = \frac{1}{(2x-1)(3x-1)}$

By the method of partial fractions, we can write

$$\frac{1}{(2x-1)(3x-1)} = \frac{2}{2x-1} + \frac{3}{3x-1}$$

Thus, $y = \frac{2}{2x-1} + \frac{3}{3x-1}$

Now, $y_n = D^n\left(\frac{2}{2x-1}\right) + D^n\left(\frac{3}{3x-1}\right)$

or, $y_n = 2\frac{(-1)^n 2^n \, n!}{(2x-1)^{n+1}} + 3\frac{(-1)^n 3^n \, n!}{(3x-1)^{n+1}}$ $\left[\because \ D^n\left(\frac{1}{ax+b}\right) = \frac{(-1)^n \, a^n n!}{(ax+b)^{n+1}}\right]$

or, $y_n = (-1)^n n! \left[\frac{2^{n+1}}{(2x-1)^{n+1}} + \frac{3^{n+1}}{(3x-1)^{n+1}}\right]$

Exercise for Reader

Find n^th^ derivative of the following functions:

i. $\sqrt{x}$

ii. $tan^{-1}\frac{x}{a}$

iii. $sin^4 x$

Lecture 6

Learning Objectives
At the end of this class, students should be able to:

- use Leibnitz's theorem
- understand the concept of Rolle's theorem
- solve the related problems

Illustration

Find the n[th] derivative of $\sin^4 x$.

Solution

Let $y = \sin^4 x$

We know that $\sin^2 x = \frac{1-\cos 2x}{2}$. Then

$$\sin^4 x = \left(\frac{1-\cos 2x}{2}\right)^2 = \frac{1}{4} - \frac{\cos 2x}{2} + \frac{\cos^2 2x}{4}$$

$$= \frac{1}{4} - \frac{\cos 2x}{2} + \frac{1}{8}(1 + \cos 4x)$$

$$= \frac{3}{8} - \frac{1}{2}\cos 2x + \frac{1}{8}\cos 4x$$

Now, $D^n(\sin^4 x) = D^n\left(\frac{3}{8}\right) - D^n\left(\frac{1}{2}\cos 2x\right) + D^n\left(\frac{1}{8}\cos 4x\right)$

$$= -\frac{1}{2}2^n \cos\left(2x + \frac{n\pi}{2}\right) + \frac{1}{8}4^n \cos\left(4x + \frac{n\pi}{2}\right)$$

Illustration

Find the n[th] derivative of $x \log\left(\frac{x-1}{x+1}\right)$.

Solution

Let $y = x \log\left(\frac{x-1}{x+1}\right)$. Then

$$y_1 = x\left(\frac{x+1}{x-1}\right) \times \frac{2}{(x+1)^2} + \log\left(\frac{x-1}{x+1}\right)$$

$$= \frac{2x}{(x-1)(x+1)} + \log(x-1) - \log(x+1)$$

$$= \frac{1}{(x-1)} + \frac{1}{(x+1)} + \log(x-1) - \log(x+1) \quad \text{[By the method of partial fractions]}$$

Again, $y_2 = -\frac{1}{(x-1)^2} - \frac{1}{(x+1)^2} + \frac{1}{(x-1)} - \frac{1}{(x+1)}$

Now, differentiating $(n-2)$ times, we get

$$y_n = (-1)^{n-2}\left[-\frac{(n-1)!}{(x-1)^n} - \frac{(n-1)!}{(x+1)^n} + \frac{(n-2)!}{(x-1)^{n-1}} - \frac{(n-2)!}{(x+1)^{n-1}}\right]$$

$$\left[\because \quad D^n\left(\frac{1}{ax+b}\right) = \frac{(-1)^n a^n n!}{(ax+b)^{n+1}}\right]$$

Leibnitz's Theorem

If *u* and *v* are any two functions of *x* such that all their desired differential coefficients exist, then the n[th] differential coefficient of their product is given by

$$D^n(uv) = (D^n u).v + n_{C_1}D^{n-1}u.Dv + n_{C_2}D^{n-2}u.D^2v + \ldots + n_{C_r}D^{n-r}u.D^r v + \ldots + u.D^n v$$

i.e.,

$$D^n(uv) = (D^n u).v + nD^{n-1}u.Dv + \frac{n(n-1)}{2!}D^{n-2}u.D^2v + \ldots + nDu.D^{n-1}v + u.D^n v$$

Illustration

If $y = \sin(m\sin^{-1}x)$, then prove that

i) $(1-x^2)y_2 - xy_1 + m^2y = 0$
ii) $(1-x^2)y_{n+2} - (2n+1)xy_{n+1} + (m^2-n^2)y = 0$

Solution

We have $y = \sin(m\sin^{-1}x)$. Then

$$y_1 = \cos(m\sin^{-1}x).\frac{m}{\sqrt{1-x^2}}$$

or, $\quad (1-x^2)y_1^2 = m^2\cos^2(m\sin^{-1}x)$

or, $\quad (1-x^2)y_1^2 = m^2 - m^2\sin^2(m\sin^{-1}x) = m^2 - m^2y^2$

$\therefore (1-x^2)y_1^2 + m^2y^2 = m^2$

Again, differentiating both sides, we get

$$2y_1y_2(1-x^2) - 2xy_1^2 + 2m^2yy_1 = 0$$

Dividing by $2y_1$, we get

$$y_2(1-x^2) - xy_1 + m^2y = 0$$

Now differentiating n times by Leibnitz's theorem, we get

$$y_{n+2}(1-x^2) + ny_{n+1}(-2x) + \frac{n(n-1)}{2!}y_n(-2) - xy_{n+1} - ny_n + m^2y_n = 0$$

or, $\quad (1-x^2)y_{n+2} - (2n+1)xy_{n+1} + (m^2-n^2)y_n = 0$

Illustration

Find the value of the nth derivative of $e^{m\sin^{-1}x}$ for $x = 0$.

Solution

Let $y = e^{m\sin^{-1}x}$ (i)

Then

$$y_1 = e^{m\sin^{-1}x}.\frac{m}{\sqrt{1-x^2}} \qquad \text{(ii)}$$

or, $(1-x^2)y_1^2 = m^2y^2$

Again, differentiating both sides, we get

$$(1-x^2)2y_1y_2 - 2xy_1^2 = 2m^2yy_1$$

Dividing by $2y_1$, we get

$$(1-x^2)y_2 - xy_1 = m^2y \qquad \text{(iii)}$$

Now differentiating n times by Leibnitz's theorem, we get

$$(1-x^2)y_{n+2} - 2nxy_{n+1} - n(n-1)y_n - xy_{n+1} - ny_n = m^2y_n$$

or, $(1-x^2)y_{n+2} - (2n+1)xy_{n+1} - (n^2+m^2)y_n = 0$

Putting $x = 0$, we get

$$y_{n+2}(0) = (n^2+m^2)y_n(0) \qquad \text{(iv)}$$

From equations (i), (ii), and (iii), we get

$$y(0) = 1,\ y_1(0) = m,\ y_2(0) = m^2$$

Putting $n = 1, 2, 3, 4$ etc. in equation (iv), we get

$$y_3(0) = (1^2+m^2)y_1(0) = m(1^2+m^2);$$
$$y_4(0) = (2^2+m^2)y_2(0) = m^2(2^2+m^2);$$
$$y_5(0) = (3^2+m^2)y_3(0) = m(1^2+m^2)(3^2+m^2);$$
$$y_6(0) = (4^2+m^2)y_4(0) = m^2(2^2+m^2)(4^2+m^2);$$

In general

$$y_n(0) = \begin{cases} m^2(2^2+m^2)(4^2+m^2)\ \ldots\ [(n-2)^2+m^2], & \text{when } n \text{ is even} \\ m(1^2+m^2)(3^2+m^2)\ \ldots\ [(n-2)^2+m^2], & \text{when } n \text{ is odd} \end{cases}$$

Rolle's Theorem

If a function $f(x)$ is

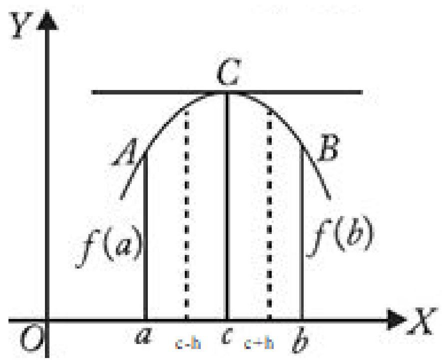
i) continuous in a closed interval $[a, b]$
ii) differentiable in the open interval (a, b) and
iii) $f(a) = f(b)$

Then there exists at least one value c of x in (a, b) such that $f'(c) = 0$.

Proof: Since $f(x)$ is continuous in $[a, b]$, $f(x)$ assumes absolute maximum and minimum values on $[a, b]$. There exists c, d in $[a, b]$ such that $f(c) = M$ (the maximum value of $f(x)$) and $f(d) = m$ (the minimum value of $f(x)$)

Case I: If $M = m$ then $f(x)$ is constant in $[a, b]$. Therefore, $f'(x) = 0$, for all x in (a, b). Thus, the theorem is true in this case.

Case II: If $M \neq m$ then at least one of them is different from $f(a)$ and $f(b)$. Let $M \neq f(a)$ & $f(b)$ and $f(c) = M$. We need to show $f'(c) = 0$ with c in (a, b).



At $x = c$,

RHD = $\lim_{h \to 0} \frac{f(c+h) - f(c)}{h}$; for $h \geq 0$

$= \lim_{h \to 0} \frac{f(c+h) - M}{h} = \frac{-\text{ve}}{+\text{ve}} \leq 0$ (i)

LHD = $\lim_{h \to 0} \frac{f(c-h) - f(c)}{-h}$

$= \lim_{h \to 0} \frac{f(c-h) - M}{-h} = \frac{-\text{ve}}{-\text{ve}} \geq 0$ (ii)

Since $f'(x)$ exists in (a, b), therefore (i) and (ii) must be equal, and this will be possible only when both (i) and (ii) are equal to zero.

Hence $f'(c) = 0$ with c in (a, b)

Case III: If $m \neq f(a)$ & $f(b)$, we can similarly show that $f'(d) = 0$ with d in (a, b). This completes the proof.

Illustration

Very Rolle's theorem for $f(x) = x^2 - 5x + 10$ in [2, 3].

Solution

i) The function $f(x) = x^2 - 5x + 10$ is a polynomial function, so it is continuous in [2, 3].

ii) Here $f'(x) = 2x - 5$, which exists for all values of x in (2, 3).

iii) Here $f(2) = 2^2 - 5 \times 2 + 10 = 4$ and $f(3) = 3^2 - 5 \times 3 + 10 = 4$.

Thus $f(2) = f(3)$

Therefore, all the conditions of Rolle's theorem are satisfied. So there exists at least one number c in $(2, 3)$ such that $f'(c) = 0$.

Now, $f'(c) = 2c - 5 = 0$

$\Rightarrow c = 5/2 \in (2, 5)$

Hence, Rolle's theorem is verified.

Illustration

Very Rolle's theorem for $f(x) = \frac{\sin x}{e^x}$ in [0, π].

Solution

i) Since the function $\sin x$ is continuous in $(-\infty, \infty)$; so, will be continuous in [0, π]. Similarly, e^x is continuous in [0, π]. Thus $f(x)$ is continuous in [0, π].

ii) Here $f'(x) = \frac{e^x.\cos x - \sin x.e^x}{(e^x)^2}$

$= \frac{\cos x - \sin x}{e^x}$, which exists for all values of x in (0, π).

iii) Here $f(0) = \frac{sin\ 0}{e^0} = \frac{0}{1} = 0$ and

$$f(\pi) = \frac{\sin \pi}{e^\pi} = \frac{0}{e^\pi} = 0$$

Thus $f(0) = f(\pi)$

Therefore, all the conditions of Rolle's theorem are satisfied. So there exists at least one number c in (0, π) such that $f'(c) = 0$.

Now, $f'(c) = \frac{\cos c - \sin c}{e^c} = 0$

$\Rightarrow \cos c - \sin c = 0$

$\Rightarrow \tan c = 1 = \tan\left(\frac{\pi}{4}\right)$

$\Rightarrow c = \pi/4 \in (0, \pi)$

Hence, Rolle's theorem is verified.

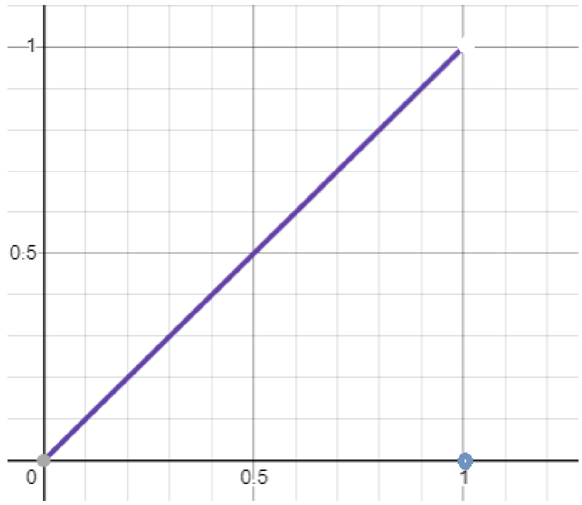
Illustration

Very Rolle's theorem for the following function on [0, 1]:

$$f(x) = \begin{cases} x & \text{if } 0 \le x < 1 \\ 0 & \text{if } x = 1 \end{cases}$$

Solution

The graph of the above function is shown below.



i) From the graph, we see that the given function is not continuous at $x = 1$ which is a point on the interval [0, 1].
ii) The given function is differentiable in (0, 1).
iii) $f(0) = f(1) = 0$

Since the first hypothesis of Rolle's theorem is not verified, so the theorem fails.

Exercise for Reader

1. If $y^{1/m} + y^{-1/m} = 2x$, show that $(x^2 - 1)y_{n+2} + (2n + 1)xy_{n+1} + (n^2 - m^2)y_n = 0$.
2. If $y = \frac{\ln x}{x}$, then $y_n = \frac{(-1)^n\, n!}{x^{n+1}}\left[\ln x - 1 - \frac{1}{2} - \frac{1}{3} - \dots - \frac{1}{n}\right]$.
3. Very Rolle's theorem for $f(x) = x(x + 3)e^{-x/2}$ in [-3, 0].
4. Very Rolle's theorem for $f(x) = \sin x$ in [-π, π].