1.4. Systems of Linear Equations.

Methods for Solving Linear Systems with Non-Singular Coefficient Matrix

– *Systems of Linear Equations (definition, solution, consistent and inconsistent systems)*
– *Matrix Method*
– *Cramer's Rule*
– *Gauss Method*
– *Gauss Method in Matrix Terms*

1.4.1. Basic Definitions

Many practical problems can be reduced to solving systems of linear equations. The main purpose of linear algebra is to find systematic methods for solving these systems.

A linear equation in n variables $x_1, x_2, \ldots, x_n$ is an equation of the form

$$a_1 x_1 + a_2 x_2 + \ldots + a_n x_n = b,$$

where $x_1, x_2, \ldots, x_n$ are *the unknowns* (i.e. quantities to be found) and a_i, $i = \overline{1,n}$, are *the coefficients* (i.e. given numbers). Also given the number b known as the *constant term* (*free member*).

! Observe that *all variables in the linear equation occur only to the first power.*

A system of linear equations or simply *a linear system* is any finite collection of linear equations.

A linear system *of* m equations in n unknowns $x_1, x_2, \ldots, x_n$ has the form:

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n = b_1, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n = b_m. \end{cases} \tag{1.7}$$

where the coefficients and constants denote known numbers.

Definition 1.12.

A system of linear equations (1.7) *is called homogeneous, if all its free members are equal to zero* $(b_1 = b_2 = \ldots = b_n = 0)$, *and non- homogeneous, if at least one of them is nonzero.*

Definition 1.13.

A solution of a linear system (1.7) *is called a finite ordered collection of numbers* $c_1, c_2, \ldots, c_n$, *which make each equation of the system* (1.7) *a true equality, when* $x_1 = c_1, x_2 = c_2, \ldots, x_n = c_n$ *are substituted in* (1.7).

The solution of a system we write as a column

$$\vec{c} = \begin{pmatrix} c_1 \\ \ldots \\ c_n \end{pmatrix}.$$

A linear system can have:

1) *exactly one solution*, for example,

$$\begin{cases} x_1 + x_2 = 1, \\ x_1 - x_2 = -1; \end{cases}$$

2) *infinitely many solutions*, for example,

$$\begin{cases} x_1 + x_2 = 0, \\ 3x_1 + 3x_2 = 0; \end{cases}$$

3) *no solutions* at all, for example,

$$\begin{cases} x_1 - x_2 = 0, \\ 2x_1 - 2x_2 = 3. \end{cases}$$

There are no other possibilities.

Definition 1.14.

A linear system is called consistent, if it has at least one solution. A system of equations which yields no solution is said to be inconsistent.

Definition 1.15.

A particular solution of a linear system is any common solution of these equations. The set of all particular solutions of a linear system is a general solution of a system.

Consider a system of linear equations (1.7). Having introduced some of the basic concepts of linear algebra, we are ready to rewrite this system in a compact form.

The matrix of the coefficients of the x_i's

$$\mathrm{A} = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix}$$

let's call *the coefficient matrix*.

The matrix of the coefficients of the x_i's and the right hand side coefficients

$$\bar{\mathrm{A}} = \left(\begin{array}{ccc|c} a_{11} & \cdots & a_{1n} & b_1 \\ \vdots & \ddots & \vdots & \vdots \\ a_{m1} & \cdots & a_{mn} & b_m \end{array}\right)$$

is called *the augmented matrix*.

Finally, if we let

$$\mathrm{X} = \begin{pmatrix} x_1 \\ \dots \\ x_n \end{pmatrix}$$

and

$$\mathrm{B} = \begin{pmatrix} b_1 \\ \dots \\ b_m \end{pmatrix},$$

then the system (1.7) can be represented in a matrix notation as

$$\begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} x_1 \\ \dots \\ x_n \end{pmatrix} = \begin{pmatrix} b_1 \\ \dots \\ b_m \end{pmatrix},$$

or

$$\mathrm{AX} = \mathrm{B}.$$

1.4.2. Linear Systems with Non-Singular Coefficient Matrix

At first we confine our attention to the linear systems, the coefficient matrix of which is non-singular:

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n = b_1, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ a_{n1}x_1 + a_{n2}x_2 + \ldots + a_{nn}x_n = b_n, \end{cases}$$

where

$$\mathrm{A} = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix}, \qquad \det \mathrm{A} \neq 0.$$

And now we consider the methods of solving such systems.

Matrix Method

Let us assume that the coefficient matrix A is a square non-singular matrix. As we remember such matrix has the inverse matrix A^{-1}.

Left-multiplying the matrix equation

$$\mathrm{AX} = \mathrm{B}$$

by the inverse matrix A^{-1}, we get

$$\mathrm{A}^{-1}\mathrm{AX} = \mathrm{A}^{-1}\mathrm{B}.$$

By definition

$$\mathrm{A}^{-1}\mathrm{A} = \mathrm{I}, \ \mathrm{I} \cdot \mathrm{X} = \mathrm{X}.$$

It leads to

$$\mathrm{X} = \mathrm{A}^{-1}\mathrm{B}.$$

We have just solve the matrix equation and obtained the unique solution.

Example

Let's solve the following linear system, using matrix method:

$$\begin{cases} x + 2y - z = 5, \\ 2x + 3y - 4z = 11, \\ 3x + y + z = 6. \end{cases}$$

❍ This system can be expressed in the matrix equation form as

$$\begin{pmatrix} 1 & 2 & -1 \\ 2 & 3 & -4 \\ 3 & 1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 5 \\ 11 \\ 6 \end{pmatrix} \Rightarrow \mathrm{AX} = \mathrm{B},$$

where

$$\mathrm{A} = \begin{pmatrix} 1 & 2 & -1 \\ 2 & 3 & -4 \\ 3 & 1 & 1 \end{pmatrix}, \quad \mathrm{B} = \begin{pmatrix} 5 \\ 11 \\ 6 \end{pmatrix}, \quad \mathrm{X} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}.$$

As

$$\det \mathrm{A} = \begin{vmatrix} 1 & 2 & -1 \\ 2 & 3 & -4 \\ 3 & 1 & 1 \end{vmatrix} = -14 \neq 0,$$

then the inverse matrix A^{-1} exists:

$$\mathrm{A}^{-1} = \frac{1}{(-14)}\begin{pmatrix} 7 & -3 & -5 \\ -14 & 4 & 2 \\ -7 & 5 & -1 \end{pmatrix}$$

(see section 1.3).
Hence,

$$\mathrm{X} = \mathrm{A}^{-1}\mathrm{B} = \frac{1}{(-14)}\begin{pmatrix} 7 & -3 & -5 \\ -14 & 4 & 2 \\ -7 & 5 & -1 \end{pmatrix}\begin{pmatrix} 5 \\ 11 \\ 6 \end{pmatrix} = \frac{1}{(-14)}\begin{pmatrix} -28 \\ -14 \\ 14 \end{pmatrix}.$$

Thus,

$$X = \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}, \quad \text{or} \quad \begin{cases} x = 2, \\ y = 1, \\ z = -1. \end{cases} \quad \bullet$$

Remark

The Matrix method is also used to solve the matrix equations.
Consider the matrix equation.

$$AX = C.$$

Suppose further that the matrix A *is invertible, i.e.* $\det A \neq 0$, *then the matrix equation has the unique solution*

$$X = A^{-1}C.$$

If we have the matrix equation

$$XB = C,\ \det B \neq 0,$$

then

$$X = CB^{-1}.$$

And the matrix equation

$$AXB = C,\ \det A \neq 0,\ \det B \neq 0,$$

has the solution

$$X = A^{-1}CB^{-1}.$$

Example

Let's solve the matrix equation:

$$\begin{pmatrix} 1 & 2 & 1 \\ 2 & 1 & -1 \\ 3 & 3 & 2 \end{pmatrix} X = \begin{pmatrix} 2 & -1 \\ 1 & 1 \\ 2 & 1 \end{pmatrix}$$

❍ We have the following matrix equation

$$AX = C,$$

where

$$A = \begin{pmatrix} 1 & 2 & 1 \\ 2 & 1 & -1 \\ 3 & 3 & 2 \end{pmatrix},$$

$$\det A = \begin{vmatrix} 1 & 2 & 1 \\ 2 & 1 & -1 \\ 3 & 3 & 2 \end{vmatrix} = 2 - 6 + 6 - 3 + 3 - 8 = -6 \neq 0,$$

and

$$C = \begin{pmatrix} 2 & -1 \\ 1 & 1 \\ 2 & 1 \end{pmatrix}.$$

Thus,

$$X = A^{-1} \cdot C = \begin{pmatrix} 1 & 2 & 1 \\ 2 & 1 & -1 \\ 3 & 3 & 2 \end{pmatrix}^{-1} \begin{pmatrix} 2 & -1 \\ 1 & 1 \\ 2 & 1 \end{pmatrix}.$$

Let's find the inverse matrix:

$$\begin{pmatrix} 1 & 2 & 1 \\ 2 & 1 & -1 \\ 3 & 3 & 2 \end{pmatrix}^{-1} = \frac{1}{(-6)} \begin{pmatrix} \begin{vmatrix} 1 & -1 \\ 3 & 2 \end{vmatrix} & -\begin{vmatrix} 2 & 1 \\ 3 & 2 \end{vmatrix} & \begin{vmatrix} 2 & 1 \\ 1 & -1 \end{vmatrix} \\ -\begin{vmatrix} 2 & -1 \\ 3 & 2 \end{vmatrix} & \begin{vmatrix} 1 & 1 \\ 3 & 2 \end{vmatrix} & -\begin{vmatrix} 1 & 1 \\ 2 & -1 \end{vmatrix} \\ \begin{vmatrix} 2 & 1 \\ 3 & 3 \end{vmatrix} & -\begin{vmatrix} 1 & 2 \\ 3 & 3 \end{vmatrix} & \begin{vmatrix} 1 & 2 \\ 2 & 1 \end{vmatrix} \end{pmatrix} =$$

$$= -\frac{1}{6} \begin{pmatrix} 5 & -1 & -3 \\ -7 & -1 & 3 \\ 3 & 3 & -3 \end{pmatrix} = \frac{1}{6} \begin{pmatrix} -5 & 1 & 3 \\ 7 & 1 & -3 \\ -3 & -3 & 3 \end{pmatrix}.$$

Therefore, we get

$$X = \frac{1}{6} \begin{pmatrix} -5 & 1 & 3 \\ 7 & 1 & -3 \\ -3 & -3 & 3 \end{pmatrix} \begin{pmatrix} 2 & -1 \\ 1 & 1 \\ 2 & 1 \end{pmatrix} = \frac{1}{6} \begin{pmatrix} -3 & 9 \\ 9 & -9 \\ -3 & 3 \end{pmatrix} = \begin{pmatrix} -1/2 & 3/2 \\ 3/2 & -3/2 \\ -1/2 & 1/2 \end{pmatrix}. \quad \bullet$$

Cramer's Rule

One of the most popular methods for solving simple systems of linear equations is *Cramer's rule* . Cramer's rule (named after Swiss mathematician *Gabriel Cramer*) describes, how to find the solution to a linear system, using *determinants*.

Gabriel Cramer

One of the useful things about Cramer's rule is that it allows us to find the value of a single variable in a linear system without having to solve the whole system.

Let us consider a linear system in a matrix form

$$\mathrm{AX} = \mathrm{B}$$

with square non-singular matrix A:

$$\det \mathrm{A} = \Delta \neq 0\,.$$

As we show above

$$\mathrm{X} = \mathrm{A}^{-1}\mathrm{B}\,,$$

or (let us remember the formula for the inverse matrix)

$$\mathrm{X} = \frac{1}{\det \mathrm{A}} \begin{pmatrix} A_{11} & A_{21} & \dots & A_{n1} \\ A_{12} & A_{22} & \dots & A_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ A_{1n} & A_{2n} & \dots & A_{nn} \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix},$$

or

$$\begin{pmatrix} x_1 \\ x_2 \\ \dots \\ x_n \end{pmatrix} = \frac{1}{\det \mathrm{A}} \begin{pmatrix} A_{11}b_1 + A_{21}b_2 + \dots + A_{n1}b_n \\ A_{12}b + A_{22}b_2 + \dots + A_{n2}b_n \\ \dots\dots\dots\dots\dots\dots\dots\dots \\ A_{1n}b_1 + A_{2n}b_2 + \dots + A_{nn}b_n \end{pmatrix}$$

It follows from this that

$$x_j = \frac{1}{\Delta}(A_{1j}b_1 + A_{2j}b_2 + \ldots + A_{nj}b_n) = \frac{1}{\Delta}\sum_{i=1}^{n} b_i A_{ij}, \quad j = \overline{1,n},$$

or we may write

$$x_j = \frac{1}{\Delta}\begin{vmatrix} a_{11} \cdots a_{1(j-1)} & b_1 & a_{1(j+1)} \cdots a_{1n} \\ a_{21} \cdots a_{2(j-1)} & b_2 & a_{2(j+1)} \cdots a_{2n} \\ \cdots\cdots\cdots & \cdots & \cdots\cdots\cdots \\ a_{n1} \cdots a_{n(j-1)} & b_n & a_{n(j+1)} \cdots a_{nn} \end{vmatrix} = \frac{\Delta_j}{\Delta}, \quad j = \overline{1,n},$$

where Δ is the determinant of a matrix A, Δ_j is the determinant of the matrix A, in which instead of its j-th column we put the column of free coefficients (column B).

Thus we obtain the *formulas of Cramer*:

$$x_j = \frac{\Delta_j}{\Delta}, \; j = \overline{1,n}.$$

Example

Let's find the solution of the linear system by Cramer's rule:

$$\begin{cases} x + 2y - z = 5, \\ 2x + 3y - 4z = 11, \\ 3x + y + z = 6. \end{cases}$$

❍ By Cramer's rule we have

$$x = \frac{\Delta_1}{\Delta}, \; y = \frac{\Delta_2}{\Delta}, \; z = \frac{\Delta_3}{\Delta},$$

$$\Delta = \begin{vmatrix} 1 & 2 & -1 \\ 2 & 3 & -4 \\ 3 & 1 & 1 \end{vmatrix} = -14 \neq 0\,, \qquad \Delta_1 = \begin{vmatrix} 5 & 2 & -1 \\ 11 & 3 & -4 \\ 6 & 1 & 1 \end{vmatrix} = -28\,,$$

$$\Delta_2 = \begin{vmatrix} 1 & 5 & -1 \\ 2 & 11 & -4 \\ 3 & 6 & 1 \end{vmatrix} = -14\,, \qquad \Delta_3 = \begin{vmatrix} 1 & 2 & 5 \\ 2 & 3 & 11 \\ 3 & 1 & 6 \end{vmatrix} = 14\,.$$

Finally,

$$x = \frac{-28}{-14} = 2\,, \quad y = \frac{-14}{-14} = 1\,, \quad z = \frac{14}{-14} = -1\,. \; \bullet$$

Remark

1. *If*

$$\det \mathrm{A} = \Delta \neq 0\,,$$

a system $\mathrm{AX} = \mathrm{B}$ *has exactly one solution.*

2. *If*

$$\det \mathrm{A} = \Delta = 0\,,$$

and at least one of Δ_j *is not equal to zero* ($\Delta_j \neq 0$), *a system has no solutions.*

3. *If*

$$\det \mathrm{A} = \Delta = 0\,,$$

and for every j $\Delta_j = 0$, *Cramer's method does not give any precise answer.*

Cramer's rule is very useful for 2×2 *systems, acceptable for* 3×3 *systems, and of doubtful use for* 4×4 *or larger systems.*

Gauss Method

Carl Gauss

Gauss method consists in the following: we perform the elementary row operations to the augmented matrix of the given system and reduce this matrix to the triangle form. The resulting system is *equivalent* to the original system (they have the same solution set), but it has a simpler form. Finally we solve the obtained system by *the method of back substitution*.

Let us consider the illustrative example now.

Example

Let's solve the linear system by Gauss method:

$$\begin{cases} x+2y-z=5, \\ 2x+3y-4z=11, \\ 3x+y+z=6. \end{cases}$$

❍ Let us reduce the augmented matrix to the upper triangle form:

$$\bar{A}=\left(\begin{array}{ccc|c} 1 & 2 & -1 & 5 \\ 2 & 3 & -4 & 11 \\ 3 & 1 & 1 & 6 \end{array}\right) \begin{array}{r} \overleftarrow{b}_1=\overleftarrow{a}_1 \\ \overleftarrow{b}_2=\overleftarrow{a}_2-2\overleftarrow{a}_1 \\ \overleftarrow{b}_3=\overleftarrow{a}_3-3\overleftarrow{a}_1 \end{array} \sim$$

$$\sim \left(\begin{array}{ccc|c} 1 & 2 & -1 & 5 \\ 0 & -1 & -2 & 1 \\ 0 & -5 & 4 & -9 \end{array}\right) \begin{array}{r} \overleftarrow{c}_1=\overleftarrow{b}_1 \\ \overleftarrow{c}_2=-\overleftarrow{b}_2 \\ \overleftarrow{c}_3=\overleftarrow{b}_3-5\overleftarrow{b}_2 \end{array} \sim \left(\begin{array}{ccc|c} 1 & 2 & -1 & 5 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & 14 & -14 \end{array}\right).$$

The later matrix corresponds to the system

$$\begin{cases} x+2y-z=5, \\ \qquad y+2z=-1, \\ \qquad\quad 14z=-14, \end{cases}$$

which is equivalent to the initial system.

The solution can be easily found, using back-substitution: the unknowns are found starting from the bottom of the system.

$$14z = -14 \Rightarrow z = -1\,,$$

$$y + 2z = -1 \Rightarrow y = -1 - 2z = 1\,,$$

$$x + 2y - z = 5 \Rightarrow x = 5 - 2y + z = 2\,.$$

Thus, we obtain the solution of the given system:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}. \qquad \bullet$$

Gauss Method in Matrix Terms

Let us solve the linear system

$$\begin{pmatrix} 1 & 2 & -1 \\ 2 & 3 & -4 \\ 3 & 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 5 \\ 11 \\ 6 \end{pmatrix}$$

or, as we may write,

$$\mathrm{AX} = \mathrm{B}$$

by *Gauss method in matrix terms*.

According to Gauss method we first have to introduce zeros in the first column in order to eliminate variable x from consideration in the last two equations. So we subtract two times the first row from the second row and three times the first row from the third row. Our transformations we write down as a matrix.

We take the Identity matrix as a basis (in our case it is the Identity matrix of the third order).

So as a result of our first transformations we get the matrix:

$$E_1 = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -3 & 0 & 1 \end{pmatrix}.$$

Further we multiply this matrix by matrix A:

$$E_1 \cdot A = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -3 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 & -1 \\ 2 & 3 & -4 \\ 3 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 & -1 \\ 0 & -1 & -2 \\ 0 & -5 & 4 \end{pmatrix} = C.$$

Then we consider the matrix C. We subtract the second row of the matrix C from the third row of this matrix five times (this transformation we write down in the second column of the Identity matrix).

We get new matrix

$$E_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -5 & 1 \end{pmatrix}$$

and multiply it by C:

$$E_2 \cdot C = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -5 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 & -1 \\ 0 & -1 & -2 \\ 0 & -5 & 4 \end{pmatrix} = \begin{pmatrix} 1 & 2 & -1 \\ 0 & -1 & -2 \\ 0 & 0 & 14 \end{pmatrix}.$$

Thus, the given matrix equation

$$AX = B$$

takes the form

$$E_2E_1AX = E_2E_1B,$$

or

$$UX = E_2E_1B, \qquad E_2E_1A \overset{\text{den}}{=} U,$$

where E_1 and E_2 are simple lower triangular matrices and

$$U = \begin{pmatrix} 1 & 2 & -1 \\ 0 & -1 & -2 \\ 0 & 0 & 14 \end{pmatrix}$$

is upper triangular matrix.

The right side of the matrix equation looks like:

$$E_2E_1B = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -3 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -5 & 1 \end{pmatrix}\begin{pmatrix} 5 \\ 11 \\ 6 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -3 & -5 & 1 \end{pmatrix}\begin{pmatrix} 5 \\ 11 \\ 6 \end{pmatrix} = \begin{pmatrix} 5 \\ 1 \\ -14 \end{pmatrix}.$$

So, we obtain

$$\begin{pmatrix} 1 & 2 & -1 \\ 0 & -1 & -2 \\ 0 & 0 & 14 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 5 \\ 1 \\ -14 \end{pmatrix},$$

and we can compute

$$X = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

by back-substitution:

$$14z = -14 \Rightarrow z = -1,$$

$$-y - 2z = 1 \Rightarrow y = -2z - 1 = 1,$$

$$x + 2y - z = 5 \Rightarrow x = 5 - 2y + z = 2.$$

And we have

$$X = \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}.$$

Remark

We may use Gauss method to solve matrix equations.

Let us demonstrate the application of this method by example.

Example

Let's solve by Gauss method the matrix equation:

$$\begin{pmatrix} 1 & 2 & 1 \\ 2 & 1 & -1 \\ 3 & 3 & 2 \end{pmatrix} X = \begin{pmatrix} 2 & -1 \\ 1 & 1 \\ 2 & 1 \end{pmatrix}.$$

❍ First we form the double-wide matrix

$$\left(\begin{array}{ccc|cc} 1 & 2 & 1 & 2 & -1 \\ 2 & 1 & -1 & 1 & 1 \\ 3 & 3 & 2 & 2 & 1 \end{array}\right).$$

We now perform elementary row operations and try to reduce the left hand half of the double-wide matrix to the Identity matrix I_3. If we succeed in doing so, then the right hand half of the double-wide matrix gives matrix X.

We have

$$\left(\begin{array}{ccc|cc} 1 & 2 & 1 & 2 & -1 \\ 2 & 1 & -1 & 1 & 1 \\ 3 & 3 & 2 & 2 & 1 \end{array}\right) \begin{array}{l} \overleftarrow{b}_1 = \overleftarrow{a}_1 \\ \overleftarrow{b}_2 = \overleftarrow{a}_2 - 2\overleftarrow{a}_1 \\ \overleftarrow{b}_3 = \overleftarrow{a}_3 - 3\overleftarrow{a}_1 \end{array} \sim$$

$$\sim \left(\begin{array}{ccc|cc} 1 & 2 & 1 & 2 & -1 \\ 0 & -3 & -3 & -3 & 3 \\ 0 & -3 & -1 & -4 & 4 \end{array}\right) \begin{array}{l} \overleftarrow{c}_1 = \overleftarrow{b}_1 \\ \overleftarrow{c}_2 = (-1/3)\overleftarrow{b}_2 \\ \overleftarrow{c}_3 = \overleftarrow{b}_3 - 3\overleftarrow{b}_2 \end{array} \sim$$

$$\sim\left(\begin{array}{ccc|cc} 1 & 2 & 1 & 2 & -1 \\ 0 & 1 & 1 & 1 & -1 \\ 0 & 0 & 2 & -1 & 1 \end{array}\right)\begin{array}{c} \vec{d}_1=\vec{c}_1-2\vec{c}_2 \\ \vec{d}_2=\vec{c}_2 \\ \vec{d}_3=1/2\vec{c}_3 \end{array}\sim$$

$$\sim\left(\begin{array}{ccc|cc} 1 & 0 & -1 & 0 & 1 \\ 0 & 1 & 1 & 1 & -1 \\ 0 & 0 & 1 & -1/2 & 1/2 \end{array}\right)\begin{array}{c} \vec{e}_1=\vec{d}_1+\vec{d}_3 \\ \vec{e}_2=\vec{d}_2-\vec{d}_3 \\ \vec{e}_3=\vec{d}_3 \end{array}\sim$$

$$\sim\left(\begin{array}{ccc|cc} 1 & 0 & 0 & -1/2 & 3/2 \\ 0 & 1 & 0 & 3/2 & -3/2 \\ 0 & 0 & 1 & -1/2 & 1/2 \end{array}\right).$$

The left hand half is the Identity matrix I_3. It follows that the right hand half of the of the double-wide matrix represents the matrix X.

Hence,

$$X=\begin{pmatrix} -1/2 & 3/2 \\ 3/2 & -3/2 \\ -1/2 & 1/2 \end{pmatrix}$$

(the same result was obtained when the given matrix equation was solved by the matrix method).●

If you decide to apply the Gauss method to the matrix equation of the form

$$XA=B,$$

then you will have to first covert the given equation to the following form

$$A^T X^T=B^T.$$