1.3. Inverse Matrix. Matrix Rank

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- *Matrix Rank*
- *Methods of finding Matrix Rank*
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1.3.1. Methods of Finding Inverse Matrix

Finding an inverse of a matrix is very important in many areas of science. For example, as we considered, decrypting a coded message uses the inverse of a matrix.

Determinant may be used to answer this problem. Indeed, let A be a square matrix of the n-th order, the determinant of which doesn't equal zero. Such matrix is called *a Non-Singular* (*Regular*) *matrix*. As we know, a matrix A^{-1} is the inverse of a matrix A if the following identities are valid:

$$\mathrm{A}\mathrm{A}^{-1} = \mathrm{A}^{-1}\mathrm{A} = \mathrm{I}\,.$$

And now we are defining the conditions of the existence of the inverse matrix. The next theorem gives us the answer.

Theorem 1.2.

A square matrix is invertible if and only if it is a non-singular matrix.

▶ $\boxed{\Rightarrow}$Let us prove, if there the inverse matrix exists, then this matrix is non-singular.

Suppose that the inverse matrix exists. Then it is carried out the equality

$$\mathrm{A}\mathrm{A}^{-1} = \mathrm{I}\,.$$

Let’s calculate a determinant of the both sides of this equality. Using the properties of a determinant we obtain

$$\det(\mathrm{A}\mathrm{A}^{-1}) = \det \mathrm{A} \cdot \det \mathrm{A}^{-1} = 1 \neq 0\,.$$

Therefore,

$$\det \mathrm{A} \neq 0\,,$$

and the matrix A is non-singular.

$\boxed{\Leftarrow}$Suppose that the matrix

$$\mathrm{A} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} = (a_{ij})$$

is non-singular:

$$\det \mathrm{A} \neq 0\,.$$

Let us show that this matrix has the inverse matrix.

For matrix A *we are constructing the matrix* C, *elements of which are the cofactors of the corresponding elements of the given matrix* A:

$$\mathrm{C} = \begin{pmatrix} A_{11} & A_{12} & \cdots & A_{1n} \\ A_{21} & A_{22} & \cdots & A_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ A_{n1} & A_{n2} & \cdots & A_{nn} \end{pmatrix} = (A_{ij})\,.$$

Then we *transpose this matrix*, finally we obtain

$$\mathrm{C}^{\mathrm{T}} = \begin{pmatrix} A_{11} & A_{21} & \cdots & A_{n1} \\ A_{12} & A_{22} & \cdots & A_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ A_{1n} & A_{2n} & \cdots & A_{nn} \end{pmatrix} = (A_{ji})\,.$$

The matrix C^{T} is called *the Adjoint matrix* of A and is denoted as $\operatorname{adj}(\mathrm{A})$.

Further, we deduce that

$$\mathrm{A}\cdot\operatorname{adj}(\mathrm{A}) = \mathrm{AC}^{\mathrm{T}} = \det\mathrm{A}\cdot\mathrm{I}.$$

Actually, let us evaluate $\mathrm{A}\cdot\mathrm{C}^{\mathrm{T}}$. We have

$$\mathrm{AC}^{\mathrm{T}} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}\begin{pmatrix} A_{11} & A_{21} & \cdots & A_{n1} \\ A_{12} & A_{22} & \cdots & A_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ A_{1n} & A_{2n} & \vdots & A_{nn} \end{pmatrix} \overset{\text{den}}{=} \mathrm{B},\ \mathrm{B} = (b_{ij}).$$

Then we get

$$b_{ij} = (a_{i1}a_{i2}...a_{in})\cdot\begin{pmatrix} A_{j1} \\ A_{j2} \\ \cdots \\ A_{jn} \end{pmatrix} = \sum_{k=1}^{n} a_{ik}A_{jk} = \begin{cases} 0, & i \neq j, \\ \det\mathrm{A}, & i = j \end{cases}.$$

Thus,

$$\mathrm{A}\cdot\operatorname{adj}(\mathrm{A}) = \begin{pmatrix} \det\mathrm{A} & 0 & \cdots & 0 \\ 0 & \det\mathrm{A} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \det\mathrm{A} \end{pmatrix} =$$

$$= \det\mathrm{A}\cdot\begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix} = \det\mathrm{A}\cdot\mathrm{I}.$$

Similarly we may prove that $\operatorname{adj}(\mathrm{A})\cdot\mathrm{A} = \det\mathrm{A}\cdot\mathrm{I}$. So we have

$$\mathrm{A}\left(\frac{1}{\det\mathrm{A}}\operatorname{adj}(\mathrm{A})\right) = \left(\frac{1}{\det\mathrm{A}}\operatorname{adj}(\mathrm{A})\right)\mathrm{A} = \mathrm{I}.$$

That’s why

$$A^{-1} = \frac{1}{\det A}\operatorname{adj}(A) = \frac{1}{\det A}\begin{pmatrix} A_{11} & A_{21} & \cdots & A_{n1} \\ A_{12} & A_{22} & \cdots & A_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ A_{1n} & A_{2n} & \cdots & A_{nn} \end{pmatrix}. \tag{1.6}$$

And the proof of the theorem is completed. ◄

Example

Let’s find the inverse of the matrix

$$A = \begin{pmatrix} 1 & 2 & -1 \\ 2 & 3 & -4 \\ 3 & 1 & 1 \end{pmatrix}.$$

❍ At first, let’s make sure, that our matrix is non-singular. So we calculate its determinant

$$\det A = \begin{vmatrix} 1 & 2 & -1 \\ 2 & 3 & -4 \\ 3 & 1 & 1 \end{vmatrix} = -14 \neq 0.$$

Thus, our matrix is invertible. The inverse matrix we are finding by formula

$$A^{-1} = \frac{1}{\det A}\begin{pmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{pmatrix}.$$

Let’s calculate the corresponding cofactors of the elements:

$$A_{11} = (-1)^2\begin{vmatrix} 3 & -4 \\ 1 & 1 \end{vmatrix} = 7, \qquad A_{21} = (-1)^3\begin{vmatrix} 2 & -1 \\ 1 & 1 \end{vmatrix} = -3,$$

$$A_{12}=(-1)^3\begin{vmatrix}2 & -4\\ 3 & 1\end{vmatrix}=-14\,,\qquad A_{22}=(-1)^4\begin{vmatrix}1 & -1\\ 3 & 1\end{vmatrix}=4\,,$$

$$A_{13}=(-1)^4\begin{vmatrix}2 & 3\\ 3 & 1\end{vmatrix}=-7\,,\qquad A_{23}=(-1)^5\begin{vmatrix}1 & 2\\ 3 & 1\end{vmatrix}=5\,,$$

$$A_{31}=(-1)^4\begin{vmatrix}2 & -1\\ 3 & -4\end{vmatrix}=-5\,,\quad A_{32}=(-1)^5\begin{vmatrix}1 & -1\\ 2 & -4\end{vmatrix}=2\,,$$

$$A_{33}=(-1)^6\begin{vmatrix}1 & 2\\ 2 & 3\end{vmatrix}=-1\,.$$

Hence,

$$\mathrm{A}^{-1}=\frac{1}{(-14)}\begin{pmatrix}7 & -3 & -5\\ -14 & 4 & 2\\ -7 & 5 & -1\end{pmatrix}=\frac{1}{14}\begin{pmatrix}-7 & 3 & 5\\ 14 & -4 & -2\\ 7 & -5 & 1\end{pmatrix}.$$

Verification:

$$\mathrm{A}^{-1}\mathrm{A}=\frac{1}{14}\begin{pmatrix}-7 & 3 & 5\\ 14 & -4 & -2\\ 7 & -5 & 1\end{pmatrix}\begin{pmatrix}1 & 2 & -1\\ 2 & 3 & -4\\ 3 & 1 & 1\end{pmatrix}=\frac{1}{14}\begin{pmatrix}14 & 0 & 0\\ 0 & 14 & 0\\ 0 & 0 & 14\end{pmatrix}=\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}=\mathrm{I}$$

and

$$\mathrm{A}\mathrm{A}^{-1}=\frac{1}{14}\begin{pmatrix}1 & 2 & -1\\ 2 & 3 & -4\\ 3 & 1 & 1\end{pmatrix}\begin{pmatrix}-7 & 3 & 5\\ 14 & -4 & -2\\ 7 & -5 & 1\end{pmatrix}=\frac{1}{14}\begin{pmatrix}14 & 0 & 0\\ 0 & 14 & 0\\ 0 & 0 & 14\end{pmatrix}=\mathrm{I}\,.$$

The product of the two matrices is indeed the identity matrix, so we're done. ●

Remark

Any singular matrix (the determinant of such matrix is equal to zero) has no an inverse matrix.

Remark

For a matrix of the second order

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

the inverse matrix A^{-1} *can be found, using the formula*

$$A^{-1} = \frac{1}{\det A}\begin{pmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{pmatrix}.$$

In other words: we swap the positions of a_{11} *and* a_{22}, *put negatives in front of* a_{21} *and* a_{12}, *and divide everything by the determinant.*

A diagonal matrix

$$D(a_{11}, a_{22}, \ldots, a_{nn}) = \begin{pmatrix} a_{11} & 0 & \cdots & 0 \\ 0 & a_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{nn} \end{pmatrix}$$

is invertible if and only if all of its diagonal elements are nonzero; in this case the inverse of the diagonal matrix is

$$D^{-1}(a_{11}, a_{22}, \ldots, a_{nn}) = \begin{pmatrix} 1/a_{11} & 0 & \cdots & 0 \\ 0 & 1/a_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1/a_{nn} \end{pmatrix}.$$

The inverse of a square matrix A *of the* n*-th order can be found by means of elementary transformations of the following double – wide matrix*

$$A \mid I = \left(\begin{array}{ccc|ccc} a_{11} & \cdots & a_{1n} & 1 & \vdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} & 0 & \cdots & 1 \end{array}\right),$$

where I *is the Identity matrix of the same* n*-th order.*

By making use of elementary row operations:

- *interchanging (swapping) rows;*
- *multiplication a row by a nonzero constant value;*
- *addition to the row another row, multiplied by a scalar;*

we have to convert the left-hand side of the double-wide matrix into the Identity matrix, i.e. transform the double-wide matrix to the form

$$\mathrm{I} \mid \mathrm{B} \ .$$

Then

$$\mathrm{B} = \mathrm{A}^{-1}.$$

Example

Let's find the inverse of the matrix , using the row transformations

$$\mathrm{A} = \begin{pmatrix} 1 & 2 & -1 \\ 2 & 3 & -4 \\ 3 & 1 & 1 \end{pmatrix}.$$

❍ Consider the double – wide matrix

$$\mathrm{A} \mid \mathrm{I} \ = \left(\begin{array}{ccc|ccc} 1 & 2 & -1 & 1 & 0 & 0 \\ 2 & 3 & -4 & 0 & 1 & 0 \\ 3 & 1 & 1 & 0 & 0 & 1 \end{array}\right) \begin{array}{c} \overleftarrow{b}_1 = \overleftarrow{a}_1 \\ \overleftarrow{b}_2 = \overleftarrow{a}_2 - 2\overleftarrow{a}_1 \\ \overleftarrow{b}_3 = \overleftarrow{a}_3 - 3\overleftarrow{a}_1 \end{array} \rightarrow$$

$$\rightarrow \left(\begin{array}{ccc|ccc} 1 & 2 & -1 & 1 & 0 & 0 \\ 0 & -1 & -2 & -2 & 1 & 0 \\ 0 & -5 & 4 & -3 & 0 & 1 \end{array}\right) \begin{array}{c} \overleftarrow{c}_1 = \overleftarrow{b}_1 \\ \overleftarrow{c}_2 = -\overleftarrow{b}_2 \\ \overleftarrow{c}_3 = \overleftarrow{b}_3 - 5\overleftarrow{b}_2 \end{array} \rightarrow$$

$$\rightarrow \left(\begin{array}{ccc|ccc} 1 & 2 & -1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 2 & -1 & 0 \\ 0 & 0 & 14 & 7 & -5 & 1 \end{array}\right) \begin{array}{c} \overleftarrow{d}_1 = \overleftarrow{c}_1 - 2\overleftarrow{c}_2 \\ \overleftarrow{d}_2 = \overleftarrow{c}_2 \\ \overleftarrow{d}_3 = (1/14)\overleftarrow{c}_3 \end{array} \rightarrow$$

$$\rightarrow \left(\begin{array}{ccc|ccc} 1 & 0 & -5 & -3 & 2 & 0 \\ 0 & 1 & 2 & 2 & -1 & 0 \\ 0 & 0 & 1 & 7/14 & -5/14 & 1/14 \end{array}\right) \begin{array}{c} \overleftarrow{e}_1 = \overleftarrow{d}_1 + 5\overleftarrow{d}_3 \\ \overleftarrow{e}_2 = \overleftarrow{d}_2 - 2\overleftarrow{d}_3 \\ \overleftarrow{e}_3 = \overleftarrow{d}_3 \end{array} \rightarrow$$

$$\to \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -7/14 & 3/14 & 5/14 \\ 0 & 1 & 0 & 1 & -4/14 & -2/14 \\ 0 & 0 & 1 & 7/14 & -5/14 & 1/14 \end{array}\right).$$

The desired form is obtained and hence,

$$A^{-1} = \begin{pmatrix} -7/14 & 3/14 & 5/14 \\ 1 & -4/14 & -2/14 \\ 7/14 & -5/14 & 1/14 \end{pmatrix} = \frac{1}{14}\begin{pmatrix} -7 & 3 & 5 \\ 14 & -4 & -2 \\ 7 & -5 & 1 \end{pmatrix}. \bullet$$

Remark

The inverse of a square matrix A we may find by means of elementary transformations using the following form of the double – wide matrix:

$$\begin{pmatrix} A \\ \hline I \end{pmatrix} = \left(\begin{array}{ccc} a_{11} & \dots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \dots & a_{nn} \\ \hline 1 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & 1 \end{array}\right).$$

But in this case we operate only with columns.

Let’s now find, using the column transformations, the inverse of our matrix

$$A = \begin{pmatrix} 1 & 2 & -1 \\ 2 & 3 & -4 \\ 3 & 1 & 1 \end{pmatrix}.$$

And now we consider such double-wide matrix

$$\begin{pmatrix} A \\ \hline I \end{pmatrix} = \left(\begin{array}{ccc} 1 & 2 & -1 \\ 2 & 3 & -4 \\ 3 & 1 & 1 \\ \hline 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right) \vdots\ \vec{b}_1 = \vec{a}_1 \quad \vec{b}_2 = \vec{a}_2 - 2\vec{a}_1 \quad \vec{b}_3 = \vec{a}_3 + \overleftarrow{a}_1\ \vdots \to$$

$$\to \left(\begin{array}{ccc} 1 & 0 & 0 \\ 2 & -1 & -2 \\ 3 & -5 & 4 \\ \hline 1 & -2 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right) \vdots\; \vec{c}_1 = \vec{b}_1 + 2\vec{b}_2 \quad \vec{c}_2 = -\vec{b}_2 \quad \vec{c}_3 = \vec{b}_3 - 2\vec{b}_2 \;\vdots \to$$

$$\to \left(\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -7 & 5 & 14 \\ \hline -3 & 2 & 5 \\ 2 & -1 & -2 \\ 0 & 0 & 1 \end{array}\right) \vdots\; \vec{d}_1 = \vec{c}_1 \quad \vec{d}_2 = \vec{c}_2 \quad \vec{d}_3 = \frac{1}{14}\vec{c}_3 \;\vdots \to$$

$$\to \left(\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -7 & 5 & 1 \\ \hline -3 & 2 & 5/14 \\ 2 & -1 & -2/14 \\ 0 & 0 & 1/14 \end{array}\right) \vdots\; \vec{e}_1 = \vec{d}_1 + 7\vec{d}_3 \quad \vec{e}_2 = \vec{d}_2 - 5\vec{d}_3 \quad \vec{e}_3 = \vec{d}_3 \;\vdots \to$$

$$\to \left(\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ \hline -7/14 & 3/14 & 5/14 \\ 1 & -4/14 & -2/14 \\ 7/14 & -5/14 & 1/14 \end{array}\right).$$

Surprisingly, but we have got the same inverse matrix again:

$$\mathrm{A}^{-1} = \frac{1}{14}\begin{pmatrix} -7 & 3 & 5 \\ 14 & -4 & -2 \\ 7 & -5 & 1 \end{pmatrix}.$$

1.3.2. Matrix Rank

Let’s consider a rectangular matrix $\mathrm{A}_{m\times n} = (a_{ij})$.
We choose in this matrix k rows and k columns $(1 \le k \le \min(m,n))$.

Definition 1.10.

A determinant of a matrix A*, elements of which lie at the intersections of the deleted* k *rows and* k *columns, is called the* k*-th order Minor of a matrix* A *and is denoted by* M_k.

For instance, one of the second order minor M_2 of a matrix

$$\mathrm{A} = \begin{array}{cccccc} & \downarrow 2\text{–d col.} & & & \downarrow 5\text{–th col.} & \\ \left(\begin{array}{c} a_{11} \\ a_{21} \\ a_{31} \\ a_{41} \end{array}\right. & \begin{array}{c} a_{12} \\ a_{22} \\ a_{32} \\ a_{42} \end{array} & \begin{array}{c} a_{13} \\ a_{23} \\ a_{33} \\ a_{43} \end{array} & \begin{array}{c} a_{14} \\ a_{24} \\ a_{34} \\ a_{44} \end{array} & \left.\begin{array}{c} a_{15} \\ a_{25} \\ a_{35} \\ a_{45} \end{array}\right) & \begin{array}{l} \leftarrow 1\text{–}st\text{ row} \\ \\ \leftarrow 3\text{–d row} \\ \\ \end{array} \end{array}$$

is

$$M_2 = \begin{vmatrix} a_{12} & a_{15} \\ a_{32} & a_{35} \end{vmatrix}$$

(we chose the first and the third rows and the second and the fifth columns).

If $s = \min(m,n)$, then the matrix A will have $1,2,\dots,s$ orders minors. Among them there exist zero and nonzero minors.

Definition 1.11.

Rank of a nonzero matrix A *is called the maximum order of its nonzero minors.*

The rank of a matrix A is usually denoted by symbol $\mathrm{r}(\mathrm{A})$ or as $\operatorname{rang}\mathrm{A}$.

Rank of the null matrix is assumed to be equal zero.

If $\mathrm{r}(\mathrm{A}) = r > 0$ *then it means that among minors of a matrix* A *there exists at least one nonzero minor of the* r*-th order, every minor of the higher order equals to zero.*

One of the possible ways of a finding of a matrix rank consists *in consecutive calculation of its minors*.

We begin with calculation of minors of ! *the first order. If all of them are equal to zero* (we have in this case a null matrix) *the rank of the matrix equals zero.*

If *among minors of the first order at least one nonzero minor exists* we pass to calculation of the minors of *the second order.* If *all minors of the second order are equal to zero* then *the rank of the matrix is one.*

Probably, among minors of the second order there is nonzero, so we pass to calculation the minors of the third order, etc.

Process of calculations is continued until or all minors of the order $k+1$ are equal to zero, or such minors do not exist. The rank of the matrix is equal to k in both cases.

Example

Let's find the rank of the matrix

$$\mathrm{A} = \begin{pmatrix} 2 & 0 & -2 & 4 \\ 3 & 2 & -4 & 6 \\ 0 & -4 & 2 & 0 \end{pmatrix}.$$

❍ The matrix has nonzero elements.
Let us choose the element in the left upper corner of the matrix

$$A = \begin{pmatrix} 2 & 0 & -2 & 4 \\ 3 & 2 & -4 & 6 \\ 0 & -4 & 2 & 0 \end{pmatrix}$$

– the minor of the first order:

$$M_1 = 2 \neq 0\,.$$

Then we try to find among minors of the second order, that contain our minor $M_1 = 2$, nonzero minor.

And we find it:

$$M_2 = \begin{vmatrix} 2 & 0 \\ 3 & 2 \end{vmatrix} = 4 \neq 0\,.$$

Then we have to calculate minors of the third order, that contain our minor $M_2 \neq 0$:

$$M_3 = \begin{vmatrix} 2 & 0 & -2 \\ 3 & 2 & -4 \\ 0 & -4 & 2 \end{vmatrix} = 0\,, \quad M_3 = \begin{vmatrix} 2 & 0 & 4 \\ 3 & 2 & 6 \\ 0 & -4 & 0 \end{vmatrix} = 0\,.$$

All minors of the third order, that contain our $M_2 \neq 0$, are equal to zero. It means that the rank of this matrix equals 2:

$$\mathrm{r}(A) = 2\,. \quad ●$$

Corollary

1. *The determinant of a square matrix* A *of the* n*-th order is equal to zero if and only if its rank is less then* n*, i.e.*

$$\det A = 0 \Leftrightarrow \mathrm{r}(A) < n\,.$$

2. *For a determinant of a square matrix* A *of the* n*-th order the following statement is true*:

$$\det A \neq 0 \Leftrightarrow \mathrm{r}(A) = n\,.$$

Nonzero minors of the order r of a matrix are called *basic minors*, the rows and columns of a basic minor are called respectively *basic rows and basic columns*.

Theorem 1.3. (*about a basic minor*). *The basic rows* (*columns*) *of a matrix* A *are linearly independent. Each another row* (*column*) *of a matrix* A *is the linear combination of its basic rows* (*columns*).

Corollary *The rank of a matrix* A *is equal to the maximum number of linearly independent rows* (columns) *of this matrix.*

The rank of a matrix can be find by more suitable method – *by applying just those row elementary operations on matrix, which was described above for finding an inverse matrix.*

Substantiation for application of this method is the following statement.

Claim 1.1. *The rank of a matrix doesn't change after applying elementary transformations on matrix.*

After applying the finite number of elementary transformations on matrix A we finally obtain *the equivalent matrix* B ($B \sim A$).

The elementary transformations on matrix reduce this matrix to the row-echelon form.

A rectangular matrix is said to be *in row-echelon form*, if it has the following three characterizations:

– *all rows, consisting entirely of zeros, are at the bottom;*

– *the leading element* (*the leftmost non-zero element of a row*) *in each nonzero row is located in a column to the right of the leading element of the row above it;*

– *all elements in a column below a leading element are zero.*

A process of reducing a matrix to a row-echelon form is known as *Gaussian elimination*.

Claim 1.2.

The number of nonzero rows of the row-echelon form of a matrix A, produced by elementary operations on A, is equal to the rank of a matrix.

And now we are illustrating the above algorithm.

Example

Let us find the rank of the matrix

$$A=\begin{pmatrix}2&0&-2&4\\3&2&-4&6\\0&-4&2&0\end{pmatrix}$$

using Gaussian elimination.

❍ By the following steps in the Gauss algorithm we find

$$A=\begin{pmatrix}2&0&-2&4\\3&2&-4&6\\0&-4&2&0\end{pmatrix}\ \vdots\ \overleftarrow{b}_2=\overleftarrow{a}_2-(3/2)\overleftarrow{a}_1\ \vdots\ \sim$$

$$\sim\begin{pmatrix}2&0&-2&4\\0&2&-1&0\\0&-4&2&0\end{pmatrix}\ \vdots\ \overleftarrow{c}_3=\overleftarrow{b}_3+2\overleftarrow{b}_2\ \vdots\ \sim\begin{pmatrix}\boxed{\begin{matrix}2&0&-2&4\\0&2&-1&0\end{matrix}}\\ \begin{matrix}0&0&0&0\end{matrix}\end{pmatrix}.$$

We obtain two nonzero rows.

Hence,

$$\mathrm{r}(A)=2. \qquad \bullet$$