1.2. Determinants

– *Definition of the Determinant*
– *Main Properties of Determinants*
– *Expansion of the Determinant in terms of rows and columns*
– *Methods of calculation of the Determinant*
– *Method of Minors of the Second Order*

Determinants appear naturally during the solution of systems of linear equations. A determinant tells us something about matrices, helps us to find the inverse of a matrix, a determinant is useful in calculus and more.

1.2.1. Definition of Determinant

Let us consider an arbitrary square matrix A of the n-th order

$$\mathrm{A} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}.$$

With each *square matrix* we can associate *a real number*

$$\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}$$

called *a Determinant*.

N.B. *Determinant exists only for square matrices*.

A determinant can be represented with the following compact notation:

$$\det \mathrm{A} = \left|\mathrm{A}\right|.$$

Number n of rows (columns) of this matrix is called *the order of a determinant*. Calculating of the determinant depends on the order of a matrix A.

For $n = 1$:

$$\left|a_{11}\right| = a_{11},$$

that is, the determinant of a matrix with one element is equal to the matrix element itself.

A determinant of the second order is called a scalar

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}.$$

The algorithm of calculation of a determinant of the second order is easy to remember when you think of a cross (schematically represented on Fig. 1.20).

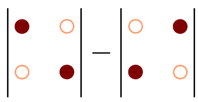

Fig. 1.20

Next, let us turn to 3×3 matrices.

For *a determinant of the third order* we obtain a number:

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} =$$

$$= a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} -$$

$$-a_{13}a_{22}a_{31} - a_{12}a_{21}a_{33} - a_{11}a_{23}a_{32}.$$

Schematically the algorithm of calculation (*Rule of Sarrus*) of a determinant of the third order is represented on Fig. 1.21.

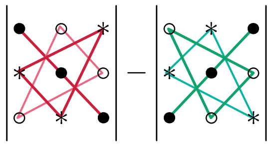

Fig. 1.21

We may state this rule most simply if we first supplement the above determinant with its first two columns repeated as below:

$$\begin{array}{|ccc|cc} a_{11} & a_{12} & a_{13} & a_{11} & a_{12} \\ a_{21} & a_{22} & a_{23} & a_{21} & a_{22} \\ a_{31} & a_{32} & a_{33} & a_{31} & a_{32} \end{array}$$

$$\begin{array}{cccccc} \swarrow & \swarrow & \swarrow & \searrow & \searrow & \searrow \\ - & - & - & + & + & + \end{array}$$

With reference to such a diagram, the rule then becomes: *add the products of the three elements on each of the diagonals sloping down to the right, and subtract from this sum the products of the three elements on each of the diagonals sloping down to the left.*

The expression for calculating the determinant of the third order can be written in a more tractable form. By factoring, we have

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}(a_{22}a_{33} - a_{23}a_{32}) + a_{12}(a_{23}a_{31} - a_{21}a_{33}) + a_{13}(a_{21}a_{32} - a_{22}a_{31}).$$

Each term in parentheses is recognized as the determinant of the second order:

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} + a_{12}\left(-\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}\right) + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}.$$

The determinants in the formula above correspond to their coefficients in the following manner:

a_{11} is the coefficient of the determinant of the matrix obtained by deleting *the first row and the first column* of the given square matrix A_3;

a_{12} is the coefficient of the negative of the determinant of the matrix obtained by deleting *the first row and the second column* of matrix A_3; and finally,

a_{13} is the coefficient of the determinant of the matrix obtained by deleting *the first row and the third column* of A_3.

In other words, the coefficients in this formula are simply the elements of the first row of the matrix A_3 (compare with the second-order determinant calculation formula).

And now we are giving the definition (inductively) of a determinant of the n-th order $(n \geq 2)$. We assume that we have defined the determinant of the order $(n-1)$.

Let's consider a square matrix A of the n-th order:

$$\mathrm{A} = \begin{array}{cc} \begin{array}{ccccccc} \phantom{a_{(i-1)1}} & & \phantom{a_{(i-1)(j-1)}} & j\text{–th col.} & \phantom{a_{(i-1)(j+1)}} & & \phantom{a_{(i-1)n}} \end{array} & \\ \begin{pmatrix} a_{11} & \cdots & a_{1(j-1)} & a_{1j} & a_{1(j+1)} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{(i-1)1} & \cdots & a_{(i-1)(j-1)} & a_{(i-1)j} & a_{(i-1)(j+1)} & \cdots & a_{(i-1)n} \\ a_{i1} & \cdots & a_{i(j-1)} & \boxed{a_{ij}} & a_{i(j+1)} & \cdots & a_{in} \\ a_{(i+1)1} & \cdots & a_{(i+1)(j-1)} & a_{(i+1)j} & a_{(i+1)(j+1)} & \cdots & a_{(i+1)n} \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{n(j-1)} & a_{nj} & a_{n(j+1)} & \cdots & a_{nn} \end{pmatrix} & \begin{array}{l} \\ \\ \\ i\text{–th row} \\ \\ \\ \\ \end{array} \end{array}$$

Let us remove *the i-th row* and *the j-th column* of this matrix. The row and the column intersect the element a_{ij}. The result is a matrix of the $(n-1)$-st order (it is known as *a Submatrix*):

$$\begin{pmatrix} a_{11} & \cdots & a_{1(j-1)} & a_{1(j+1)} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ a_{(i-1)1} & \cdots & a_{(i-1)(j-1)} & a_{(i-1)(j+1)} & \cdots & a_{(i-1)n} \\ a_{(i+1)1} & \cdots & a_{(i+1)(j-1)} & a_{(i+1)(j+1)} & \cdots & a_{(i+1)n} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{n(j-1)} & a_{n(j+1)} & \cdots & a_{nn} \end{pmatrix}.$$

The determinant of a submatrix is called the *Minor of the element* a_{ij} and is denoted by M_{ij}, i.e.

$$M_{ij} = \begin{vmatrix} a_{11} & \cdots & a_{1(j-1)} & a_{1(j+1)} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ a_{(i-1)1} & \cdots & a_{(i-1)(j-1)} & a_{(i-1)(j+1)} & \cdots & a_{(i-1)n} \\ a_{(i+1)1} & \cdots & a_{(i+1)(j-1)} & a_{(i+1)(j+1)} & \cdots & a_{(i+1)n} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{n(j-1)} & a_{n(j+1)} & \cdots & a_{nn} \end{vmatrix}.$$

We define the term

$$A_{ij} = (-1)^{i+j} M_{ij},$$

as *the Cofactor* (*Algebraic Addition*) *of the element* a_{ij}.

Definition 1.8.

A determinant of the n*-th order* $(n \geq 2)$ *of a matrix* A *is called a scalar, which is equal to the sum of products of all elements of the first row of a matrix* A *by their corresponding cofactors*:

$$\det \mathrm{A} = \sum_{k=1}^{n} a_{1k} A_{1k}.$$

In other words, we evaluate a determinant of a matrix using the expansion by cofactors along the first row.

For example,

$$\begin{vmatrix} 2 & 5 & 0 & 2 \\ 3 & -1 & 2 & 1 \\ 4 & 0 & 1 & -2 \\ 3 & 1 & 0 & 1 \end{vmatrix} = 2 \cdot A_{11} + 5 \cdot A_{12} + 0 \cdot A_{13} + 2 \cdot A_{14}.$$

Now, the cofactors of the elements in the first row of the given matrix are

$$A_{11} = (-1)^{1+1} \cdot \begin{vmatrix} -1 & 2 & 1 \\ 0 & 1 & -2 \\ 1 & 0 & 1 \end{vmatrix} = +(-1-4-1) = -6,$$

$$A_{12} = (-1)^{1+2} \cdot \begin{vmatrix} 3 & 2 & 1 \\ 4 & 1 & -2 \\ 3 & 0 & 1 \end{vmatrix} = -(3-12-3-8) = 20,$$

$$A_{13} = (-1)^{1+3} \cdot \begin{vmatrix} 3 & -1 & 1 \\ 4 & 0 & -2 \\ 3 & 1 & 1 \end{vmatrix} = 4+6+6+4 = 20,$$

$$A_{14} = (-1)^{1+4} \cdot \begin{vmatrix} 3 & -1 & 2 \\ 4 & 0 & 1 \\ 3 & 1 & 0 \end{vmatrix} = -(8-3-3) = -2.$$

And therefore

$$\begin{vmatrix} 2 & 5 & 0 & 2 \\ 3 & -1 & 2 & 1 \\ 4 & 0 & 1 & -2 \\ 3 & 1 & 0 & 1 \end{vmatrix} = 2 \cdot (-6) + 5 \cdot 20 + 0 \cdot 20 + 2 \cdot (-2) = 84.$$

1.2.2. Properties of Determinants

Now we are going to consider some of the many properties of determinants which will help us in future to compute them efficiently, and they are interesting in applications.

Property 1

A determinant of the transpose of a matrix A *is equal to a determinant of a given matrix* A:

$$\det \mathrm{A}^{\mathrm{T}} = \det \mathrm{A}.$$

Validity of this property is established by direct calculation of determinants.

For example,

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}, \quad \begin{vmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}.$$

This property means that any techniques which are developed for rows may be also applied to columns.

Property 2

If each element of a row (column) can be expressed as binomial, the determinant can be written as the sum of two determinants.

For example, for (2×2) matrix

$$\begin{vmatrix} a'_{11} + a''_{11} & a'_{12} + a''_{12} \\ a_{21} & a_{22} \end{vmatrix} = \begin{vmatrix} a'_{11} & a'_{12} \\ a_{21} & a_{22} \end{vmatrix} + \begin{vmatrix} a''_{11} & a''_{12} \\ a_{21} & a_{22} \end{vmatrix}.$$

Property 3

If all elements of a row (column) are multiplied by a scalar the result is that a determinant is multiplied by this constant value.

For example,

$$\begin{vmatrix} ka_{11} & ka_{12} \\ a_{21} & a_{22} \end{vmatrix} = k\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}.$$

This means that the common factor of elements of a row (column) can be taken out.

If two rows (columns) of a determinant are interchanged, the sign of a determinant is reversed.

Property 4

Indeed,

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21},$$

$$\begin{vmatrix} a_{21} & a_{22} \\ a_{11} & a_{12} \end{vmatrix} = a_{21}a_{12} - a_{11}a_{22} = -(a_{11}a_{22} - a_{12}a_{21}) = -\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}.$$

Properties 1- 4 are basic properties of a determinant, that reveal its nature.

The next properties are logical consequences of the four major.

Property 5

The determinant of a matrix is equal to zero:

1) *if any row (column) contains all zero elements;*

2) *if any two rows (colums) are identical;*

3) *if two rows (colums) are propotional to each other.*

The sum of the products of the elements of any row (column) by the cofactors of the elements of another row (column) is equal to zero:

$$\sum_{j=1}^{n} a_{ij} A_{kj} = 0, i \neq k.$$

Property 6

Property 7

If to any row (column) there is added the corresponding elements of any other row (column) multiplied by a constant factor k, the value of the determinant is unchanged.

For example,

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} + ka_{11} & a_{22} + ka_{12} \end{vmatrix} = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} + \begin{vmatrix} a_{11} & a_{12} \\ ka_{11} & ka_{12} \end{vmatrix} =$$

$$= \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} + k \begin{vmatrix} a_{11} & a_{12} \\ a_{11} & a_{12} \end{vmatrix} = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} + 0 = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}.$$

The determinant of a product of two square matrices is equal to the product of the determinants of these matrices:

$$\det(\mathbf{AB}) = \det \mathbf{A} \cdot \det \mathbf{B}.$$

Property 8

Property 9

The determinant of an upper (lower) triangular matrix A is equal to the product of its diagonal elements:

$$\det \mathrm{A} = a_{11} a_{22} \cdot \ldots \cdot a_{nn} = \prod_{i=1}^{n} a_{ii}.$$

The determinant of the Identity Matrix equals the unity.

1.2.3. Some Methods of Calculation of Determinants

We will further mean by the rows and the columns of a determinant the elements of the rows and the columns of the corresponding matrix.

Let us consider now some methods of calculation of the determinant of the n-th order.

Expansion of the Determinant in Terms of Rows and Columns

As we know the determinant is equal to the sum of products of the elements of the first row by their cofactors:

$$\det \mathrm{A} = \sum_{k=1}^{n} a_{1k} A_{1k} \, .$$

The right hand side of this equality is called *the cofactor expansion of the determinant along the first row*.

It was proved that for a square matrix A is valid *the cofactor expansion of the determinant along the i-th row* $(1 \le i \le n)$:

$$\det \mathrm{A} = \sum_{k=1}^{n} a_{ik} A_{ik}$$

and *the cofactor expansion of the determinant along the j-th column* $(1 \le j \le n)$:

$$\det \mathrm{A} = \sum_{k=1}^{n} a_{kj} A_{kj} \, .$$

Thus, the determinant of a matrix A *is equal to the sum of the products of all elements of any row (column) of* A *by their corresponding cofactors.*

For example,

$$\begin{vmatrix} 2 & 5 & 0 & 2 \\ 3 & -1 & 2 & 1 \\ 4 & 0 & 1 & -2 \\ 3 & 1 & 0 & 1 \end{vmatrix} = 0 \cdot A_{13} + 2 \cdot A_{23} + 1 \cdot A_{33} + 0 \cdot A_{43}\,.$$

Here it is convenient to use the determinant expansion into elements of the third column because *this column contains more zeros* (thus we minimize the calculations).

We have:

$$A_{23} = (-1)^{2+3} \cdot \begin{vmatrix} 2 & 5 & 2 \\ 4 & 0 & -2 \\ 3 & 1 & 1 \end{vmatrix} = -(8 - 30 + 4 - 20) = 38\,,$$

$$A_{33} = (-1)^{3+3} \cdot \begin{vmatrix} 2 & 5 & 2 \\ 3 & -1 & 1 \\ 3 & 1 & 1 \end{vmatrix} = +(-2 + 15 + 6 + 6 - 2 - 15) = 8\,.$$

And we obtain

$$\begin{vmatrix} 2 & 5 & 0 & 2 \\ 3 & -1 & 2 & 1 \\ 4 & 0 & 1 & -2 \\ 3 & 1 & 0 & 1 \end{vmatrix} = 2 \cdot 38 + 1 \cdot 8 = 84\,.$$

The greatest source of error is forgetting to take the factor $(-1)^{i+j}$ *into account during the expansion.*

Reduction to Triangle Matrix

We start with the following definition.

Definition 1.9.

Elementary transformations on matrices (*elementary row operations*) *are called:*

I. *interchanging of rows* (*columns*);

II. *multiplication of a row* (*column*) *by a nonzero number;*

III. *addition to a row* (*column*) *another row* (*column*), *multiplied by a scalar.*

Method of reduction to the triangle matrix consists in the following: using the elementary transformations on matrix, remembering properties of determinants, we reduce given matrix to the triangle form.

As we know *the determinant of the triangle matrix is equal to the product of the diagonal elements.*

Let's calculate our well-known determinant

$$\begin{vmatrix} 2 & 5 & 0 & 2 \\ 3 & -1 & 2 & 1 \\ 4 & 0 & 1 & -2 \\ 3 & 1 & 0 & 1 \end{vmatrix},$$

reducing given matrix to the triangle form.

To indicate which operation is being used in the process we will write the following shorthand notation. For example, $\overleftarrow{b}_2 = 3\overleftarrow{a}_2$ represents the row operation of type (II), where each element of the second row is replaced by 3 that element. Similar interpretations we will give for types (I) and (III) operations.

So, we have:

$$\begin{vmatrix} 2 & 5 & 0 & 2 \\ 3 & -1 & 2 & 1 \\ 4 & 0 & 1 & -2 \\ 3 & 1 & 0 & 1 \end{vmatrix} \begin{array}{c} \overleftarrow{b}_1 = \overleftarrow{a}_1 \\ \overleftarrow{b}_2 = \overleftarrow{a}_2 - (3/2)\,\overleftarrow{a}_1 \\ \overleftarrow{b}_3 = \overleftarrow{a}_3 - 2\overleftarrow{a}_1 \\ \overleftarrow{b}_4 = \overleftarrow{a}_4 - \overleftarrow{a}_2 \end{array} =$$

(*let's rewrite the first row and apply row operation of type* (III) *three times to eliminate* 3, 4, 3 *in the first column*)

$$= \begin{vmatrix} 2 & 5 & 0 & 2 \\ 0 & -17/2 & 2 & -2 \\ 0 & -10 & 1 & -6 \\ 0 & 2 & -2 & 0 \end{vmatrix} \begin{array}{c} \overleftarrow{c}_1 = \overleftarrow{b}_1 \\ \overleftarrow{c}_2 = \overleftarrow{b}_4 \\ \overleftarrow{c}_3 = \overleftarrow{b}_3 \\ \overleftarrow{c}_4 = \overleftarrow{b}_2 \end{array} =$$

(*we rewrite the first and the third rows, permute the second and the fourth row*)

$$= -\begin{vmatrix} 2 & 5 & 0 & 2 \\ 0 & 2 & -2 & 0 \\ 0 & -10 & 1 & -6 \\ 0 & -17/2 & 2 & -2 \end{vmatrix} \begin{array}{c} \overleftarrow{d}_1 = \overleftarrow{c}_1 \\ \overleftarrow{d}_2 = \overleftarrow{c}_2 \\ \overleftarrow{d}_3 = \overleftarrow{c}_3 + 5\overleftarrow{c}_2 \\ \overleftarrow{d}_4 = \overleftarrow{c}_4 + (17/4)\,\overleftarrow{c}_2 \end{array} =$$

(*rewrite the first and the second rows, apply row operation of type* (III) *twice to eliminate* (-10), $(-17/2)$ *in the second column*)

$$= -\begin{vmatrix} 2 & 5 & 0 & 2 \\ 0 & 2 & -2 & 0 \\ 0 & 0 & -9 & -6 \\ 0 & 0 & -13/2 & -2 \end{vmatrix} \begin{array}{c} \overleftarrow{e}_1 = \overleftarrow{d}_1 \\ \overleftarrow{e}_2 = \overleftarrow{d}_2 \\ \overleftarrow{e}_3 = \overleftarrow{d}_3 \\ \overleftarrow{e}_4 = \overleftarrow{d}_4 - (13/18)\,\overleftarrow{d}_3 \end{array} =$$

(*let's rewrite the first, the second and the third rows, and apply row operation of type* (III) *to eliminate the last element* $(-13/2)$ *in the third column*)

$$= -\begin{vmatrix} 2 & 5 & 0 & 2 \\ 0 & 2 & -2 & 0 \\ 0 & 0 & -9 & -6 \\ 0 & 0 & 0 & 7/3 \end{vmatrix} = -\left(2 \cdot 2 \cdot (-9) \cdot \frac{7}{3}\right) = 84\,.$$

Method of Minors of Second Order

According to the method of minors of the second order we express the given determinant of the n-th order in terms of the determinant of the $(n-1)$-st order. The elements of the new determinant are the determinants of the second order.

If an element of a given determinant, standing in the top left corner (i.e. a_{11}) is non-zero, this formula has the form

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \dots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \dots & a_{nn} \end{vmatrix} = \frac{1}{(a_{11})^{n-2}} \begin{vmatrix} \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} & \begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix} & \dots & \begin{vmatrix} a_{11} & a_{1n} \\ a_{21} & a_{2n} \end{vmatrix} \\ \begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix} & \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} & \dots & \begin{vmatrix} a_{11} & a_{1n} \\ a_{31} & a_{3n} \end{vmatrix} \\ \vdots & \vdots & \ddots & \vdots \\ \begin{vmatrix} a_{11} & a_{12} \\ a_{n1} & a_{n2} \end{vmatrix} & \begin{vmatrix} a_{11} & a_{13} \\ a_{n1} & a_{n3} \end{vmatrix} & \dots & \begin{vmatrix} a_{11} & a_{1n} \\ a_{n1} & a_{nn} \end{vmatrix} \end{vmatrix}.$$

The determinant elements on the right-hand side of the equality are the determinants of the second order. They are called *the minors of the second order*.

Hence, by successive application of this formula, we obtain the determinant of the second order.

▶ Let us prove this formula.

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \dots & a_{nn} \end{vmatrix} \begin{array}{c} \overleftarrow{b}_2 = \overleftarrow{a}_2 - \dfrac{a_{21}}{a_{11}} \overleftarrow{a}_1 \\ \dots\dots\dots\dots\dots\dots \\ \overleftarrow{b}_n = \overleftarrow{a}_n - \dfrac{a_{n1}}{a_{11}} \overleftarrow{a}_1 \end{array} =$$

$$
= \begin{vmatrix}
a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\
0 & a_{22} - \dfrac{a_{21}}{a_{11}} a_{12} & a_{23} - \dfrac{a_{21}}{a_{11}} a_{13} & \cdots & a_{2n} - \dfrac{a_{21}}{a_{11}} a_{1n} \\
0 & a_{32} - \dfrac{a_{31}}{a_{11}} a_{12} & a_{33} - \dfrac{a_{31}}{a_{11}} a_{13} & \cdots & a_{3n} - \dfrac{a_{31}}{a_{11}} a_{1n} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & a_{n2} - \dfrac{a_{n1}}{a_{11}} a_{12} & a_{n3} - \dfrac{a_{n1}}{a_{11}} a_{13} & \cdots & a_{nn} - \dfrac{a_{n1}}{a_{11}} a_{1n}
\end{vmatrix} =
$$

(*then we use the expansion by the elements of the first column; we get the determinant of the* $(n-1)$*-st order*)

$$
= a_{11} \begin{vmatrix}
a_{22} - \dfrac{a_{21}}{a_{11}} a_{12} & a_{23} - \dfrac{a_{21}}{a_{11}} a_{13} & \cdots & a_{2n} - \dfrac{a_{21}}{a_{11}} a_{1n} \\
a_{32} - \dfrac{a_{31}}{a_{11}} a_{12} & a_{33} - \dfrac{a_{31}}{a_{11}} a_{13} & \cdots & a_{3n} - \dfrac{a_{31}}{a_{11}} a_{1n} \\
\vdots & \vdots & \ddots & \vdots \\
a_{n2} - \dfrac{a_{n1}}{a_{11}} a_{12} & a_{n3} - \dfrac{a_{n1}}{a_{11}} a_{13} & \cdots & a_{nn} - \dfrac{a_{n1}}{a_{11}} a_{1n}
\end{vmatrix} =
$$

$$
= \frac{a_{11}}{(a_{11})^{n-1}} \begin{vmatrix}
a_{22}a_{11} - a_{21}a_{12} & a_{23}a_{11} - a_{21}a_{13} & \cdots & a_{2n}a_{11} - a_{21}a_{1n} \\
a_{32}a_{11} - a_{31}a_{12} & a_{33}a_{11} - a_{31}a_{13} & \cdots & a_{3n}a_{11} - a_{31}a_{1n} \\
\vdots & \vdots & \ddots & \vdots \\
a_{n2}a_{11} - a_{n1}a_{12} & a_{n3}a_{11} - a_{n1}a_{13} & \cdots & a_{nn}a_{11} - a_{n1}a_{1n}
\end{vmatrix} =
$$

$$
= \frac{1}{(a_{11})^{n-2}} \begin{vmatrix}
\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} & \begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix} & \cdots & \begin{vmatrix} a_{11} & a_{1n} \\ a_{21} & a_{2n} \end{vmatrix} \\
\begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix} & \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} & \cdots & \begin{vmatrix} a_{11} & a_{1n} \\ a_{31} & a_{3n} \end{vmatrix} \\
\vdots & \vdots & \ddots & \vdots \\
\begin{vmatrix} a_{11} & a_{12} \\ a_{n1} & a_{n2} \end{vmatrix} & \begin{vmatrix} a_{11} & a_{13} \\ a_{n1} & a_{n3} \end{vmatrix} & \cdots & \begin{vmatrix} a_{11} & a_{1n} \\ a_{n1} & a_{nn} \end{vmatrix}
\end{vmatrix}. \blacktriangleleft
$$

For example, let's calculate our determinant

$$\begin{vmatrix} 2 & 5 & 0 & 2 \\ 3 & -1 & 2 & 1 \\ 4 & 0 & 1 & -2 \\ 3 & 1 & 0 & 1 \end{vmatrix}$$

by method of the minors of the second order.

We have: $a_{11} = 2 \neq 0$, $n = 4$.

That's why

$$\begin{vmatrix} 2 & 5 & 0 & 2 \\ 3 & -1 & 2 & 1 \\ 4 & 0 & 1 & -2 \\ 3 & 1 & 0 & 1 \end{vmatrix} = \frac{1}{2^2} \begin{vmatrix} \begin{vmatrix} 2 & 5 \\ 3 & -1 \end{vmatrix} & \begin{vmatrix} 2 & 0 \\ 3 & 2 \end{vmatrix} & \begin{vmatrix} 2 & 2 \\ 3 & 1 \end{vmatrix} \\ \begin{vmatrix} 2 & 5 \\ 4 & 0 \end{vmatrix} & \begin{vmatrix} 2 & 0 \\ 4 & 1 \end{vmatrix} & \begin{vmatrix} 2 & 2 \\ 4 & -2 \end{vmatrix} \\ \begin{vmatrix} 2 & 5 \\ 3 & 1 \end{vmatrix} & \begin{vmatrix} 2 & 0 \\ 3 & 0 \end{vmatrix} & \begin{vmatrix} 2 & 2 \\ 3 & 1 \end{vmatrix} \end{vmatrix} =$$

$$= \frac{1}{4} \begin{vmatrix} -17 & 4 & -4 \\ -20 & 2 & -12 \\ -13 & 0 & -4 \end{vmatrix} = \frac{1}{4(-17)} \begin{vmatrix} \begin{vmatrix} -17 & 4 \\ -20 & 2 \end{vmatrix} & \begin{vmatrix} -17 & -4 \\ -20 & -12 \end{vmatrix} \\ \begin{vmatrix} -17 & 4 \\ -13 & 0 \end{vmatrix} & \begin{vmatrix} -17 & -4 \\ -13 & -4 \end{vmatrix} \end{vmatrix} =$$

$$= -\frac{1}{68} \begin{vmatrix} 46 & 124 \\ 52 & 16 \end{vmatrix} = 84\,.$$

The determinant of a matrix has an interesting geometric interpretation.

Let's consider a square with vertexes in the points with coordinates: $O(0;0)$, $A(0;1)$, $B(1;1)$, $C(1;0)$ (Fig. 1.20).

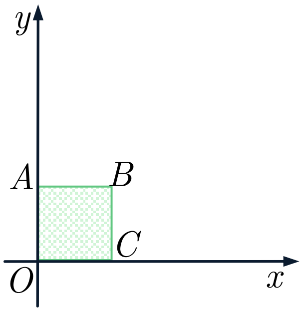

Fig. 1.20

We'll create a vertex matrix (see lecture 2):

$$\begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix}.$$

Then we multiply the vertex matrix from the left by the matrix

$$\mathrm{A} = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}.$$

We get

$$\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 3 & 2 \\ 0 & 2 & 3 & 1 \end{pmatrix}.$$

As we see, we obtain the vertex matrix of the parallelogram $OA'B'C'$ (Fig.1.21 a)).

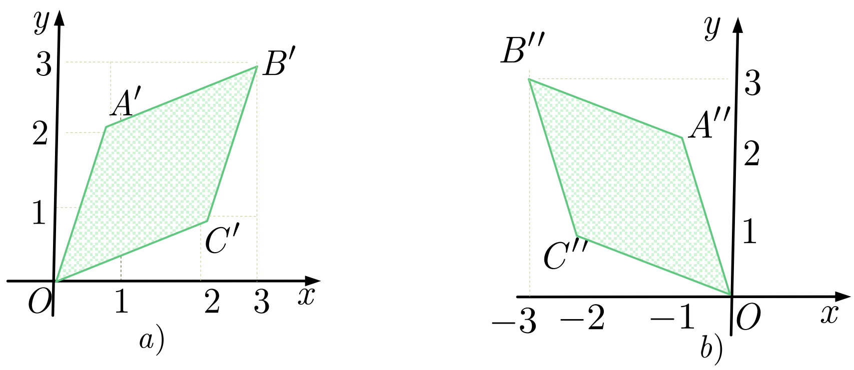

Fig. 1.21

And if we multiply our given vertex matrix from the left by the matrix

$$\mathrm{B}=\begin{pmatrix}-2 & -1\\ 1 & 2\end{pmatrix},$$

we have

$$\begin{pmatrix}-2 & -1\\ 1 & 2\end{pmatrix}\begin{pmatrix}0 & 0 & 1 & 1\\ 0 & 1 & 1 & 0\end{pmatrix}=\begin{pmatrix}0 & -1 & -3 & -2\\ 0 & 2 & 3 & 1\end{pmatrix}.$$

And this matrix represents the vertex matrix of the parallelogram $OA''B''C''$ (see Fig.1.21 b)).

The area of the parallelogram $OA'B'C'$, as can be seen from the Fig. 1.21 a), is 3:

$$S_{OA'B'C'}=3^2-2\cdot 1^2-4\cdot\frac{1}{2}\cdot 2\cdot 1=3.$$

It is easy to verify that area of the parallelogram $OA''B''C''$ is also equal to 3.

And now let’s calculate the determinants of the matrices A and B:

$$\det\mathrm{A}=\begin{vmatrix}2 & 1\\ 1 & 2\end{vmatrix}=3;$$

$$\det\mathrm{B}=\begin{vmatrix}-2 & -1\\ 1 & 2\end{vmatrix}=-3\Rightarrow\left|\det\mathrm{B}\right|=3.$$

Thus in 2D, *the absolute value of the determinant is equal to the area of a parallelogram.*