We continue to consider matrices.

A very useful operation in matrix algebra is that of transposition.

Transposition of Matrices

The operation that interchanges a row into a column and vice versa, is called transposition (is defined as T):

$$a_{11} \quad \dots \quad a_{1n} \xrightarrow{\mathrm{T}} \begin{pmatrix} a_{11} \\ \dots \\ a_{1n} \end{pmatrix}, \qquad \begin{pmatrix} a_{11} \\ \dots \\ a_{n1} \end{pmatrix} \xrightarrow{\mathrm{T}} a_{11} \quad \dots \quad a_{n1} \ .$$

For example,

$$\begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix} \xrightarrow{\mathrm{T}} 1 \ \ -2 \ \ 0 \ .$$

Definition 1.6.

The transpose of a matrix A:

$$\mathrm{A} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$$

is a matrix A^{T} *which rows are the columns of a matrix* A:

$$\mathrm{A}^{\mathrm{T}} = \begin{pmatrix} a_{11} & a_{21} & \cdots & a_{m1} \\ a_{12} & a_{22} & \cdots & a_{m2} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1n} & a_{2n} & \cdots & a_{mn} \end{pmatrix}.$$

This implies that A^{T} *will have its dimensions reversed when compared with* A.

For example, since

$$A_{2\times 3} = \begin{pmatrix} 1 & 0 & -3 \\ 2 & -5 & 7 \end{pmatrix},$$

then its transpose will be

$$A^{T}_{3\times 2} = \begin{pmatrix} 1 & 2 \\ 0 & -5 \\ -3 & 7 \end{pmatrix}.$$

The following result lists of a matrix transpose properties:

1) $(A^T)^T = A$;

2) $(A + B)^T = A^T + B^T$;

3) $(\alpha A)^T = \alpha A^T$;

4) $(AB)^T = B^T A^T$;

5) $(D)^T = D$ *for any diagonal matrix* D.

6) $\mathrm{tr}(A^T) = \mathrm{tr}A$ *for any square matrix* A.

Note carefully that the property 4) indicates that the transpose of the product is the product of the transposes *in reverse order.*

If we have linear matrix expression

$$Y = A \cdot X + B,$$

then

$$Y^T = X^T A^T + B^T.$$

Properties 2) and 4) extend to any finite sum or product of matrices.

For example,

$$(A + B + C)^T = A^T + B^T + C^T;$$

$$(A \cdot B \cdot C)^T = C^T \cdot B^T \cdot A^T.$$

A transpose of a matrix makes it easy to multiply matrices visually. We'll demonstrate this method on an example.

Let's consider matrices:

$$\mathrm{A} = \begin{pmatrix} -1 & 3 & 0 \\ 0 & 1 & 7 \end{pmatrix} \text{ and } \mathrm{B} = \begin{pmatrix} 1 & 0 & 3 \\ -5 & 2 & 0 \\ -1 & 4 & -3 \end{pmatrix}.$$

We need to find the product $\mathrm{A}\cdot\mathrm{B}$ (we have already multiplied these matrices earlier).

At first let us transpose the matrix A:

$$\mathrm{A}^{\mathrm{T}} = \begin{pmatrix} -1 & 0 \\ 3 & 1 \\ 0 & 7 \end{pmatrix}.$$

Then we find:

$$\begin{pmatrix} \boxed{\begin{matrix} -1 \\ 3 \\ 0 \end{matrix}} & \begin{matrix} 0 \\ 1 \\ 7 \end{matrix} \end{pmatrix} \vdots \begin{pmatrix} \boxed{\begin{matrix} 1 \\ -5 \\ -1 \end{matrix}} & \begin{matrix} 0 & 3 \\ 2 & 0 \\ 4 & -3 \end{matrix} \end{pmatrix} =$$

$$= \left(\begin{array}{c:c:c} \boxed{(-1)\cdot 1 + 3\cdot(-5) + 0\cdot(-1)} & (-1)\cdot 0 + 3\cdot 2 + 0\cdot 4 & (-1)\cdot 3 + 3\cdot 0 + 0\cdot(-3) \\ 0\cdot 1 + 1\cdot(-5) + 7\cdot(-1) & 0\cdot 0 + 1\cdot 2 + 7\cdot 4 & 0\cdot 3 + 1\cdot 0 + 7\cdot(-3) \end{array}\right).$$

Hence, we obtain

$$\mathrm{A}\cdot\mathrm{B} = \begin{pmatrix} \boxed{-16} & 6 & -3 \\ -12 & 30 & -21 \end{pmatrix}.$$

The elements of the matrix-product are obtained as all possible mutual products of the columns of the left (transpose) matrix and the columns of the right matrix. The place of each element of matrix $\mathrm{A}\cdot\mathrm{B}$ is determined according to the rule: *the column number of the left matrix* A^{T} *is assigned as the first subscript, and the column number of the right matrix* B *is the second subscript.*

Let’s consider another examples:

$$(a_{11}\ \ a_{12})\cdot\begin{pmatrix} b_{11}\\ b_{21}\end{pmatrix}=\begin{pmatrix} a_{11}\\ a_{12}\end{pmatrix}^{\mathrm{T}}\cdot\begin{pmatrix} b_{11}\\ b_{21}\end{pmatrix}=(a_{11}\cdot b_{11}+a_{12}\cdot b_{21});$$

$$\begin{pmatrix} b_{11}\\ b_{21}\end{pmatrix}\cdot(a_{11}\ \ a_{12})=(b_{11}\ \ b_{21})^{\mathrm{T}}\cdot(a_{11}\ \ a_{12})=\begin{pmatrix} b_{11}\cdot a_{11} & b_{11}\cdot a_{12}\\ b_{21}\cdot a_{11} & b_{21}\cdot a_{12}\end{pmatrix}.$$

Square matrix A is called *a Symmetric matrix* if

$$\mathrm{A}^{\mathrm{T}}=\mathrm{A}$$

(✓ *this implies that* $a_{ij}=a_{ji}$ *for all* i *and* j).

For example, the following matrices

$$\begin{pmatrix} 1 & 3\\ 3 & -2\end{pmatrix},\quad \begin{pmatrix} 2 & 1 & -2\\ 1 & 0 & 7\\ -2 & 7 & 5\end{pmatrix}$$

are symmetric matrices, since each matrix is equal to its own transpose.

Square matrix A is called *a Skew-Symmetric matrix* if

$$\mathrm{A}^{\mathrm{T}}=-\mathrm{A}$$

(✓ *all diagonal elements equal zero in this case*).

For example, the matrix

$$\mathrm{A}=\begin{pmatrix} 0 & 1 & -2\\ -1 & 0 & -7\\ 2 & 7 & 0\end{pmatrix}$$

is the skew-symmetric matrix since

$$\mathrm{A}^{\mathrm{T}}=\begin{pmatrix} 0 & -1 & 2\\ 1 & 0 & 7\\ -2 & -7 & 0\end{pmatrix}=-\begin{pmatrix} 0 & 1 & -2\\ -1 & 0 & -7\\ 2 & 7 & 0\end{pmatrix}=-\mathrm{A}.$$

Suppose A *and* B *are symmetric matrices with the same order, and* α *is any scalar.*
Then the following results hold:

1) A^{T} *is symmetric matrix;*

2) $\mathrm{A}+\mathrm{B}$ *and* $\mathrm{A}-\mathrm{B}$ *are symmetric matrices;*

3) $\alpha\mathrm{A}$ *is symmetric matrix.*

It is not true, in general, that the product of symmetric matrices is symmetric matrix. To see why this is so, let A and B be symmetric matrices of the same order. Then it follows from the property 4) of the transpose matrix and the symmetry of A and B that

$$(\mathrm{AB})^{\mathrm{T}}=\mathrm{B}^{\mathrm{T}}\mathrm{A}^{\mathrm{T}}=\mathrm{BA}.$$

Thus, $(\mathrm{AB})^{\mathrm{T}}=\mathrm{AB}$, if and only if $\mathrm{AB}=\mathrm{BA}$, that is, if and only if matrices A and B commute.

Matrix products of the form AA^{T} and $\mathrm{A}^{\mathrm{T}}\mathrm{A}$ arise in a variety of applications. If A is an $(m\times n)$ matrix, then A^{T} is, as we know, an $(n\times m)$ matrix, so the products AA^{T} and $\mathrm{A}^{\mathrm{T}}\mathrm{A}$ are both *square matrices* — the matrix AA^{T} has size $(m\times m)$, and the matrix $\mathrm{A}^{\mathrm{T}}\mathrm{A}$ is the square matrix of the n-th order.

N.B. *The products* AA^{T} *and* $\mathrm{A}^{\mathrm{T}}\mathrm{A}$ *are always symmetric matrices.*

We have that

$$(\mathrm{AA}^{\mathrm{T}})^{\mathrm{T}}=(\mathrm{A}^{\mathrm{T}})^{\mathrm{T}}\mathrm{A}^{\mathrm{T}}=\mathrm{AA}^{\mathrm{T}},$$

it means that the product AA^{T} is a symmetric matrix.

Similarly, we can prove that the product $\mathrm{A}^{\mathrm{T}}\mathrm{A}$ is a symmetric matrix.

Remark

N.B. Let us consider the column-matrix

$$\vec{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}.$$

Then we get

$$\vec{x}^{\mathrm{T}} \cdot \vec{x} = \begin{pmatrix} x & y & z \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} = x^2 + y^2 + z^2$$

– *dot* (*inner*) *product* (Fig. 1.5 (blue square)), and

$$\vec{x} \cdot \vec{x}^{\mathrm{T}} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \cdot \begin{pmatrix} x & y & z \end{pmatrix} = \begin{pmatrix} x^2 & xy & xz \\ yx & y^2 & yz \\ zx & zy & z^2 \end{pmatrix}$$

– *outer product* (see Fig. 1.5 (pink square)).

As we can see, *the dot product is the trace of the outer product.*

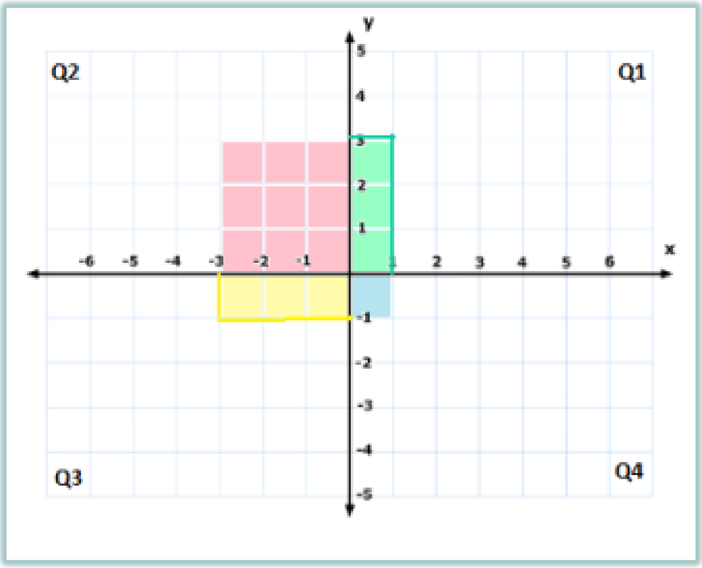

Fig. 1.5

Inverse Matrix

For matrices, there is no such thing as division. You can add, subtract, and multiply matrices, but you cannot divide them. It is possible to construct an analogue of division for matrices — multiplication by the inverse of matrix.

Definition 1.7.

The matrix A^{-1} *is called to be an inverse of a square matrix* A *if such relations are valid*

$$AA^{-1} = A^{-1}A = I,$$

where I *is the Identity matrix.*

N.B. *Only square matrices possibly have an inverse. An inverse matrix has the same size as the matrix of which it is an inverse. Not all matrices have inverses.*

When a matrix A *has an inverse, it is said to be the Invertible matrix.*

Theorem 1.1. *If a matrix* A *is invertible, then the inverse matrix* A^{-1} *is unique.*

► Assume matrices A_1^{-1} and A_2^{-1} are both inverses of A.
Let's show that

$$A_1^{-1} = A_2^{-1}.$$

Indeed,

$$A_1^{-1} = A_1^{-1} \cdot I = A_1^{-1}(AA_2^{-1}) = (A_1^{-1}A)A_2^{-1} = I \cdot A_2^{-1} = A_2^{-1}.$$

So the inverse is unique since any two inverses coincide. ◄

Suppose A *and* B *are invertible matrices. Then the following results hold*:

1) $(A^{-1})^{-1} = A$;

2) $(A^{-1})^{n} = (A^{n})^{-1}, \forall n \in \mathbb{N}$;

3) $(AB)^{-1} = B^{-1}A^{-1}$;

4) $(A^{-1})^{T} = (A^{T})^{-1}$;

5) $(\alpha A)^{-1} = \dfrac{1}{\alpha} A^{-1}$.

► Let us prove, for instance, property 1. Let's multiply the equality

$$A^{-1}(A^{-1})^{-1} = I$$

by A on the left. We get:

$$A \cdot A^{-1}(A^{-1})^{-1} = A \cdot I \Rightarrow (A \cdot A^{-1})(A^{-1})^{-1} = A \Rightarrow I \cdot (A^{-1})^{-1} = A.$$

So, we have

$$(A^{-1})^{-1} = A.$$

Let us prove the property 3:

$$(AB) \cdot (B^{-1}A^{-1}) = A \cdot (BB^{-1}) \cdot A^{-1} = A \cdot I \cdot A^{-1} = A \cdot A^{-1} = I.$$

Similarly, we can show that

$$(B^{-1}A^{-1}) \cdot (AB) = I.$$

The above is indeed true if

$$B^{-1}A^{-1} = (AB)^{-1}.$$

It means that

$$(AB)^{-1} = B^{-1}A^{-1}.$$

This property can be generalized to three or more invertible matrices, for example:

$$(ABC)^{-1} = C^{-1}B^{-1}A^{-1}. \blacktriangleleft$$

Let us consider some applications of matrices and operations with them.

Matrices and Digital Images [6]

The images on internet pages and the photos are examples of digital images. It is possible to represent this kind of image using matrices. For example, the image of the “Cat” (Fig. 1.6)

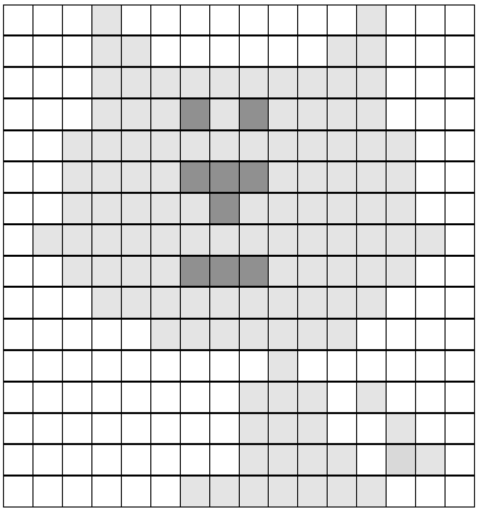

Fig. 1.6

can be represented by 16×16 matrix A (Fig. 1.7)

$$A=\begin{pmatrix}
0&0&0&1&0&0&0&0&0&0&0&0&1&0&0&0\\
0&0&0&1&1&0&0&0&0&0&0&1&1&0&0&0\\
0&0&0&1&1&1&1&1&1&1&1&1&1&0&0&0\\
0&0&0&1&1&1&2&1&2&1&1&1&1&0&0&0\\
0&0&1&1&1&1&1&1&1&1&1&1&1&1&0&0\\
0&0&1&1&1&1&2&2&2&1&1&1&1&1&0&0\\
0&0&1&1&1&1&1&2&1&1&1&1&1&1&0&0\\
0&1&1&1&1&1&1&1&1&1&1&1&1&1&1&0\\
0&0&1&1&1&1&2&2&2&1&1&1&1&1&0&0\\
0&0&0&1&1&1&1&1&1&1&1&1&1&0&0&0\\
0&0&0&0&0&1&1&1&1&1&1&1&1&0&0&0\\
0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&1&1&1&0&1&0&0&0\\
0&0&0&0&0&0&0&0&1&1&1&0&0&1&0&0\\
0&0&0&0&0&0&0&0&1&1&1&1&0&1&1&0\\
0&0&0&0&0&0&1&1&1&1&1&1&1&0&0&0
\end{pmatrix}$$

Fig. 1.7

which elements are the numbers 0, 1 and 2.

These numbers specify the color of each pixel (Fig. 1.8).

⬜	░	▓	⬛
0	1	2	3
White	Light Gray	Dark Gray	Black

Fig. 1.8

Each element of the matrix A determines the intensity of the corresponding pixel. In order to increase the contrast of the image “Cat”, it is necessary to add the matrix B (Fig. 1.9) to the matrix A.

$$B=\begin{pmatrix}
2 & 2 & 2 & -1 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & -1 & 2 & 2 & 2\\
2 & 2 & 2 & -1 & -1 & 2 & 2 & 2 & 2 & 2 & 2 & -1 & -1 & 2 & 2 & 2\\
2 & 2 & 2 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & 2 & 2 & 2\\
2 & 2 & 2 & -1 & -1 & -1 & 1 & -1 & 1 & -1 & -1 & -1 & -1 & 2 & 2 & 2\\
2 & 2 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & 2 & 2\\
2 & 2 & -1 & -1 & -1 & -1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 & -1 & 2 & 2\\
2 & 2 & -1 & -1 & -1 & -1 & -1 & 1 & -1 & -1 & -1 & -1 & -1 & -1 & 2 & 2\\
2 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & 2\\
2 & 2 & -1 & -1 & -1 & -1 & 0 & 0 & 0 & -1 & -1 & -1 & -1 & -1 & 2 & 2\\
2 & 2 & 2 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & 2 & 2 & 2\\
2 & 2 & 2 & 2 & 2 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & 2 & 2 & 2\\
2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & -1 & 2 & 2 & 2 & 2 & 2 & 2\\
2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & -1 & -1 & -1 & 2 & -1 & 2 & 2 & 2\\
2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & -1 & -1 & -1 & 2 & 2 & -1 & 2 & 2\\
2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & -1 & -1 & -1 & -1 & 2 & -1 & -1 & 2\\
2 & 2 & 2 & 2 & 2 & 2 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & 2 & 2 & 2
\end{pmatrix}$$

Fig. 1.9

We get the matrix $A+B$ (Fig. 1.10)

$$A+B=\begin{pmatrix}
2 & 2 & 2 & 0 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 0 & 2 & 2 & 2\\
2 & 2 & 2 & 0 & 0 & 2 & 2 & 2 & 2 & 2 & 2 & 0 & 0 & 2 & 2 & 2\\
2 & 2 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 2 & 2\\
2 & 2 & 2 & 0 & 0 & 0 & 3 & 0 & 3 & 0 & 0 & 0 & 0 & 2 & 2 & 2\\
2 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 2\\
2 & 2 & 0 & 0 & 0 & 0 & 3 & 3 & 3 & 0 & 0 & 0 & 0 & 0 & 2 & 2\\
2 & 2 & 0 & 0 & 0 & 0 & 0 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 2\\
2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2\\
2 & 2 & 0 & 0 & 0 & 0 & 2 & 2 & 2 & 0 & 0 & 0 & 0 & 0 & 2 & 2\\
2 & 2 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 2 & 2\\
2 & 2 & 2 & 2 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 2 & 2\\
2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 0 & 2 & 2 & 2 & 2 & 2 & 2\\
2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 0 & 0 & 0 & 2 & 0 & 2 & 2 & 2\\
2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 0 & 0 & 0 & 2 & 2 & 0 & 2 & 2\\
2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 2\\
2 & 2 & 2 & 2 & 2 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 2 & 2
\end{pmatrix},$$

Fig. 1.10

which describes the new image “Cat” (Fig.1.11).

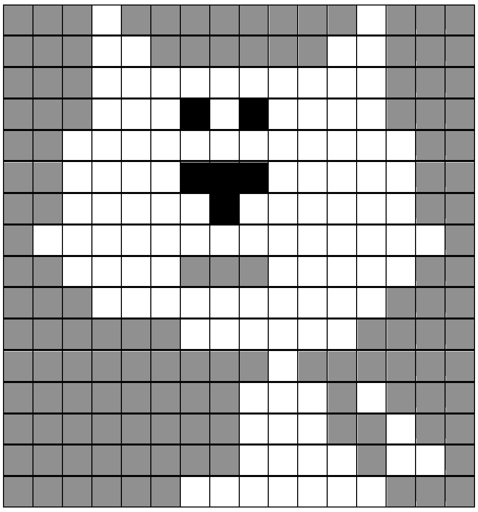

Fig. 1.11

Image (Fig. 1.12) corresponds to the transposed matrix of A+B.

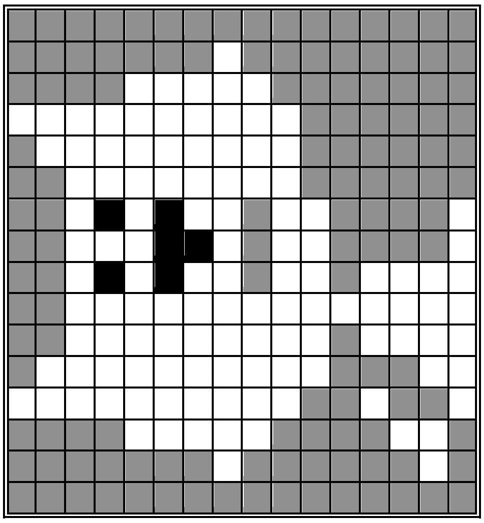

Fig. 1.12

Matrices and Encryption [6]

Data encryption has become a necessity with the rise of sensitive data being stored and transmitted via computers.

Let us demonstrate a simple encryption. The message we want to encrypt is "*Make haste slowly*". Each letter in the message is assigned a numerical value. For our encryption we are using the letters of the alphabet (Fig. 1.13).

A	B	C	D	E	F	G	H	I	J	K	L	M	N
1	2	3	4	5	6	7	8	9	10	11	12	13	14

O	P	Q	R	S	T	U	V	W	X	Y	Z	SPACE
15	16	17	18	19	20	21	22	23	24	25	26	0

Fig. 1.13

So we have:

m – 13, a – 1, k – 11, e – 5; h – 8, a – 1, s – 19, t – 20, e – 5;

s – 19, l – 12, o – 15, w – 23, l – 12, y –25.

And our message we are writing as a string of numbers:

13, 1, 11, 5, 0, 8, 1, 19, 20, 5, 0, 19, 12, 15, 23, 12, 25.

The most vital component in encryption is a key matrix which is used to encrypt the messages, and its inverse is used to decrypt the encoded messages. It is important that a key matrix is kept secret between the message senders and intended recipients.

We are using in our case a 3×3 key matrix for the encryption. Our key matrix K will be, for example:

$$K = \begin{pmatrix} 1 & 2 & -1 \\ 2 & 3 & -4 \\ 3 & 1 & 1 \end{pmatrix}.$$

However, not just any matrix can be used as a key. First, the matrix must be invertible.

Then we split our message into chunks of three letters, because the key matrix has size 3×3:

$$A=\begin{pmatrix}13 & 5 & 1 & 5 & 12 & 12\\ 1 & 0 & 19 & 0 & 15 & 25\\ 11 & 8 & 20 & 19 & 23 & 0\end{pmatrix}.$$

To encode the text we multiply the matrix A by the matrix K on the left (matrices A and K must be conformable for multiplication):

$$K\cdot A=\begin{pmatrix}1 & 2 & -1\\ 2 & 3 & -4\\ 3 & 1 & 1\end{pmatrix}\cdot\begin{pmatrix}13 & 5 & 1 & 5 & 12 & 12\\ 1 & 0 & 19 & 0 & 15 & 25\\ 11 & 8 & 20 & 19 & 23 & 0\end{pmatrix}=$$

$$=\begin{pmatrix}4 & -3 & 19 & -14 & 19 & 62\\ -15 & -22 & -21 & -66 & -23 & 99\\ 51 & 23 & 42 & 34 & 74 & 61\end{pmatrix}.$$

The message is sent as the numbers in the rows of the matrix $K\cdot A$:

4, –15, 51, –3, –22, 23, 19, –21, 42, –14, –66, 34, 19, –23, 74, 62, 99, 61.

To decode the message we have to multiply the matrix $K\cdot A$ by the matrix K^{-1} on the left (in the section 1.3 we will show how to compute this inverse matrix):

$$K^{-1}=\frac{1}{14}\begin{pmatrix}-7 & 3 & 5\\ 14 & -4 & -2\\ 7 & -5 & 1\end{pmatrix};$$

$$\frac{1}{14}\begin{pmatrix}-7 & 3 & 5\\ 14 & -4 & -2\\ 7 & -5 & 1\end{pmatrix}\begin{pmatrix}4 & -3 & 19 & -14 & 19 & 62\\ -15 & -22 & -21 & -66 & -23 & 99\\ 51 & 23 & 42 & 34 & 74 & 61\end{pmatrix}=$$

$$=\frac{1}{14}\begin{pmatrix}182 & 70 & 14 & 70 & 168 & 168\\ 14 & 0 & 266 & 0 & 210 & 350\\ 154 & 112 & 280 & 266 & 322 & 0\end{pmatrix}=\begin{pmatrix}13 & 5 & 1 & 5 & 12 & 12\\ 1 & 0 & 19 & 0 & 15 & 25\\ 11 & 8 & 20 & 19 & 23 & 0\end{pmatrix}=A\,.$$

And receiver will read:

13, 1, 11, 5, 0, 8, 1, 19, 20, 5, 0, 19, 12, 15, 23, 12, 25,

that means

“MAKE HASTE SLOWLY“.

Matrices and Transformations

Any image on a plane could be altered by using different transformations. The most common types of transformations are (see Fig. 1.14): *translation* (we slide a figure in any direction); *dilation* (we enlarge or reduce a figure); *reflection* (we flip a figure over a line) and *rotation* (we rotate a figure a certain degree around a point).

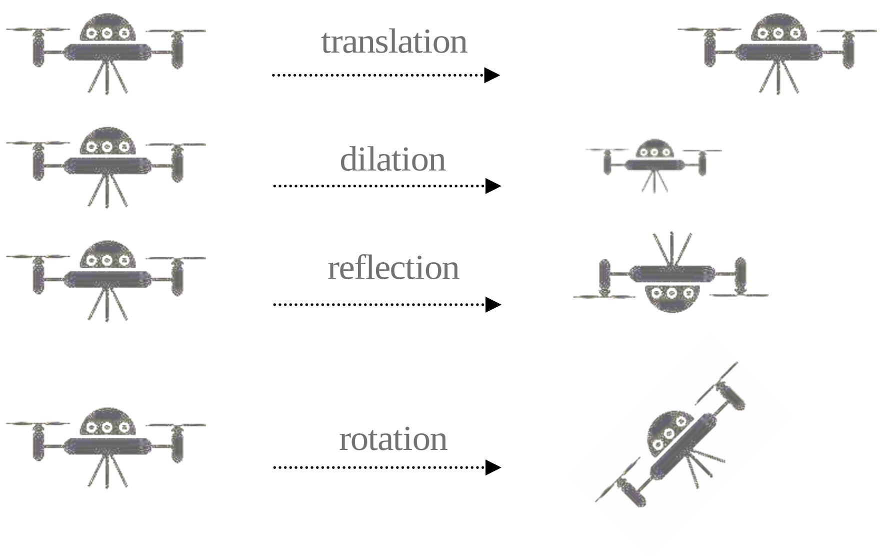

Fig. 1.14.

Let us consider a matrix which is a representation of some figure. For example, we have a square with vertexes in the points with coordinates: $(2;0)$, $(2;4)$, $(6;4)$, $(6;0)$ (Fig. 1.15).

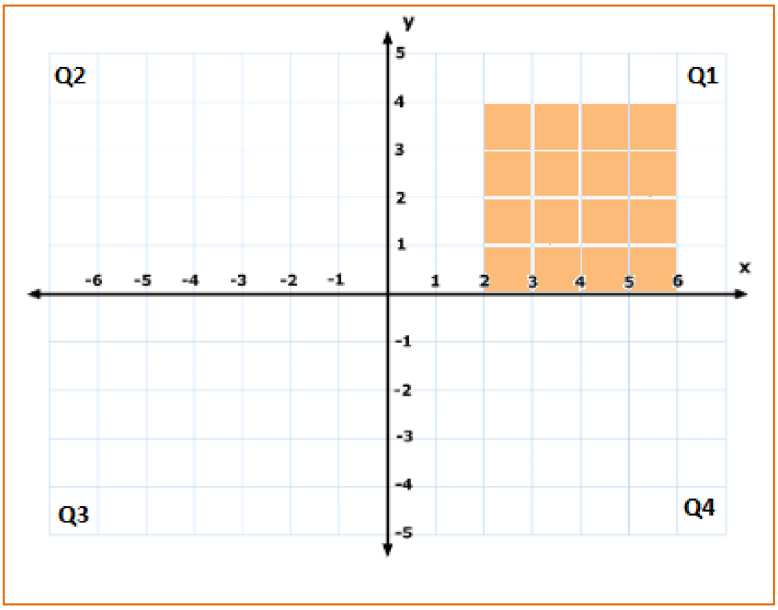

Fig. 1.15

Then we create vertex matrix:

$$\begin{pmatrix} 2 & 2 & 6 & 6 \\ 0 & 4 & 4 & 0 \end{pmatrix}.$$

If we want to translate figure we simply add to each element of the matrix necessary constant.

For example, we want *to translate* all vertexes of our square three units left. The transformation can be achieved as follows:

$$\begin{pmatrix} 2 & 2 & 6 & 6 \\ 0 & 4 & 4 & 0 \end{pmatrix} = \begin{pmatrix} 2-3 & 2-3 & 6-3 & 6-3 \\ 0-3 & 4-3 & 4-3 & 0-3 \end{pmatrix} = \begin{pmatrix} -1 & -1 & 3 & 3 \\ -3 & 1 & 1 & -3 \end{pmatrix}.$$

Here is the result of our transformation (Fig. 1.16).

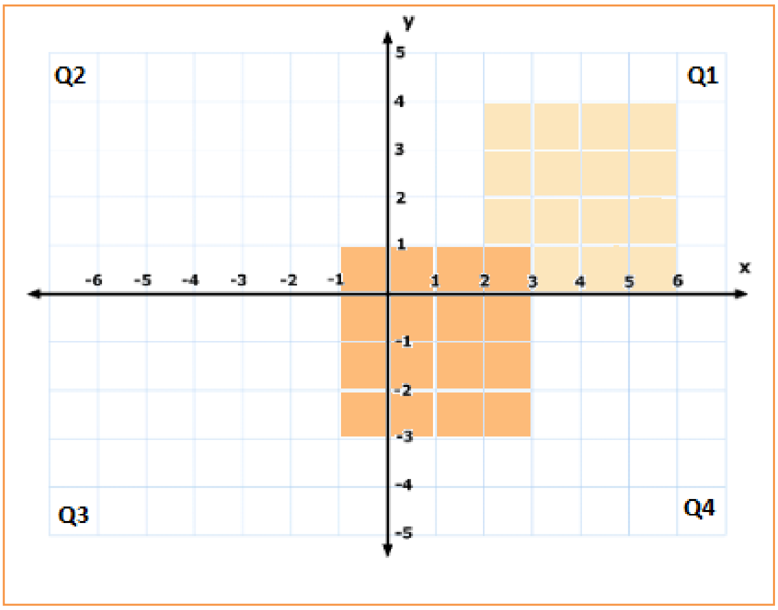

Fig.1.16

If we want *to dilate* a figure we simply multiply each element of the given matrix by the scale factor we want to dilate with.

For example, we need to halve our square. We get (Fig.1.17):

$$\frac{1}{2} \cdot \begin{pmatrix} 2 & 2 & 6 & 6 \\ 0 & 4 & 4 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 3 & 3 \\ 0 & 2 & 2 & 0 \end{pmatrix}.$$

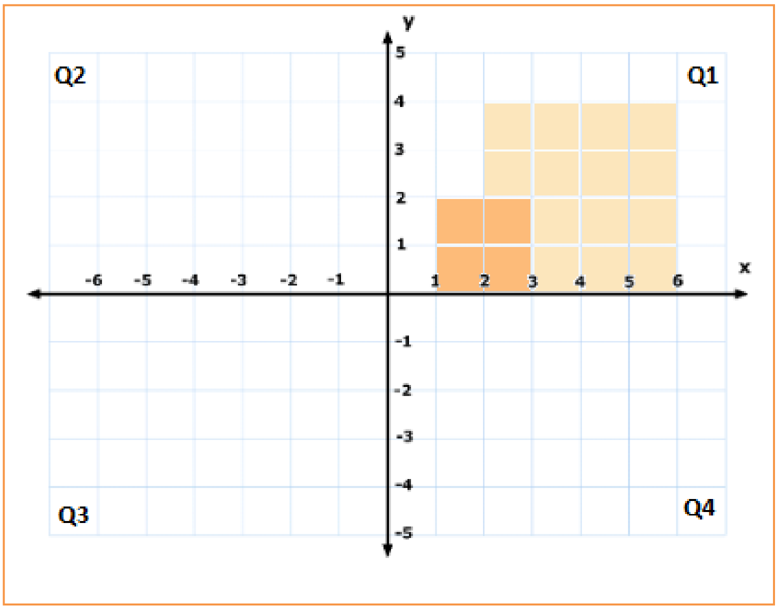

Fig. 1.17

When we want to create *a reflection* image we multiply the given matrix of our figure by what is called *a reflection matrix*. The most common reflection matrices are:

for a reflection in the x-axis

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix};$$

for a reflection in the y-axis

$$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix};$$

for a reflection in the origin

$$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Suppose we want to reflect our square in the y- axis. As the result of this transformation we get (see Fig. 1.18):

$$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 2 & 2 & 6 & 6 \\ 0 & 4 & 4 & 0 \end{pmatrix} = \begin{pmatrix} -2 & -2 & -6 & -6 \\ 0 & 4 & 4 & 0 \end{pmatrix}.$$

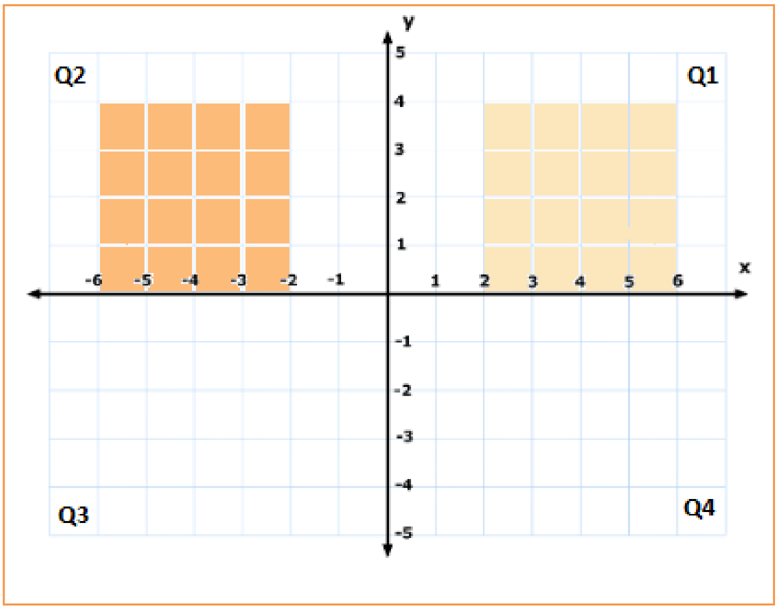

Fig. 1.18

If we want *to rotate* a figure either clockwise or counterclockwise, we multiply the given matrix by matrix below:

$$\begin{pmatrix} \cos\varphi & \sin\varphi \\ -\sin\varphi & \cos\varphi \end{pmatrix}$$

(rotation by φ in a clockwise direction),
or

$$\begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix}$$

(rotation by φ in a counterclockwise direction).

Suppose that we want to rotate the square counterclockwise about the origin by the angle $\varphi = 180^{\circ}$. The matrix required to carry out such transformation is

$$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

(compare with the reflection in the origin).

Thus, we have

$$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 2 & 2 & 6 & 6 \\ 0 & 4 & 4 & 0 \end{pmatrix} = \begin{pmatrix} -2 & -2 & -6 & -6 \\ 0 & -4 & -4 & 0 \end{pmatrix}.$$

Here is the result of this transformation (Fig.1.19).

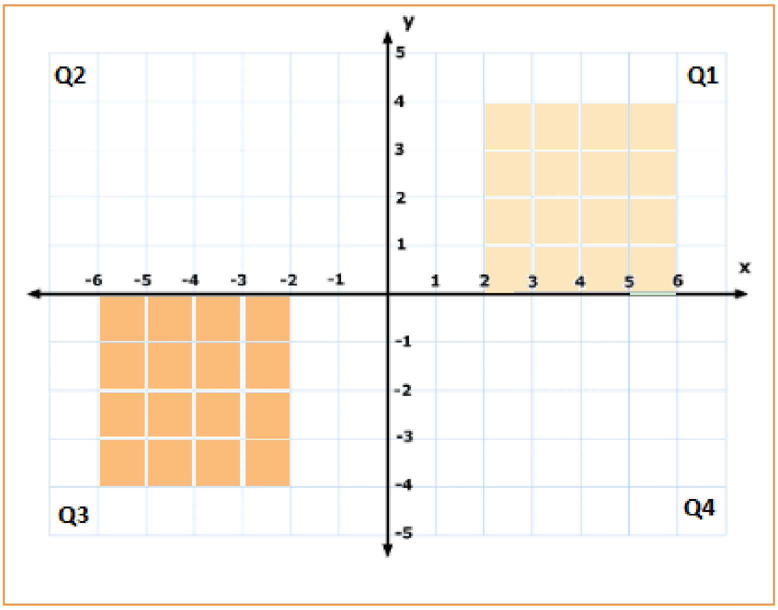

Fig. 1.19

Another applications of matrices you may see in [2].