General Relativity

We now discuss how the techniques of the theory of manifolds can be used to formulate General Relativity. We will make the following basic assumptions:

1. the set M of all possible events in the universe is a manifold of dimension four,
2. on M there is a Lorentzian metric g_{ab},
3. any pointlike test-object with a mass $m > 0$ follows a timelike curve $\xi : I \to M$,
4. in the case of free fall, i.e. in the absence of forces other than gravity, ξ is a geodesic.

From now on we will restrict attention to manifolds of dimension four and for convenience we will also choose physical units such that $c = 1$.

We have dropped the assumption that $M = \mathbb{R}^4$, which is difficult to verify experimentally, but we still assume that M locally looks like $\mathbb{R}^4$, in the sense that it is a 4-dimensional manifold. Using this formulation, all charts are treated on an equal footing and there are no preferred choices of coordinates. In other words coordinates have no intrinsic physical meaning.

We have also replaced the rather rigid structure of η_{ab}, that was of such fundamental importance in Minkowski space, by a Lorentzian metric g_{ab}. This metric may vary from point to point and in some sense we may think of it as the gravitational field. It certainly determines the geodesics and the timelike geodesics are the same for any object, regardless of its mass (weak equivalence principle).

Of course we have yet to supply an equation of motion for the physical field g_{ab}. Because the gravitational field depends on the mass distribution of the matter in the universe, the equation of motion should also depend on this matter.

Kinematics of General Relativity

In this section we will supply some kinematical aspects of General Relativity, describing how observers can measure properties of particles. We will also investigate how the framework that we have developed so far can implement the strong equivalence principle, that the outcome of any local experiment in a freely falling laboratory is independent of the initial position and velocity of the laboratory. In addition we will see how the metric g_{ab} can encode gravity.

Let us now consider an object, e.g. a particle, following a timelike trajectory $\xi : I \to M$ in a Lorentzian manifold M. Our first purpose is to define notions of future and past on M. At each point $p \in M$, the tangent space T_pM is isomorphic to Minkowski space, so it has two light cones and we can choose one of these to be the future and the other the past. We can make such a choice for any $p \in M$, but it is not obvious that we can make this choice in a way that depends smoothly (or continuously) on the base point $p \in M$. Such a smooth dependence is desirable in order to avoid pathological situations, where a smooth timelike curve suddenly switches from being future pointing to past pointing. For this reason we will only consider spacetimes which are time-oriented in the following sense:

Definition 10.1 *A spacetime (M, g_{ab}) is called* time-orientable *when there exists a vector field T^a on M which is timelike at every point of M. Such a vector field determines a* time-orientation*: a timelike vector $v^a \in T_pM$ is called future pointing if it is in the same light cone as T^a (i.e. if $v^aT_a < 0$) and past pointing otherwise ($v^aT_a > 0$).*

A spacetime is called time-oriented *if it is time-orientable and a time-orientation has been chosen.*

Not every spacetime is time-orientable. However, when a spacetime is time-orientable, then there are many timelike vector fields T^a that determine the same time-orientation. Mathematically speaking, a time-orientation is an equivalence class of timelike vector fields, where two such vector fields T^a and T'^a are equivalent if their vectors lie in the same light cone at every point $p \in M$. Because M is connected and T^a and T'^a are both timelike, it suffices to verify this condition at one point. It follows that a time-orientable Lorentzian manifold has exactly two time-orientations. We will always assume that a choice of time-orientation can and has been made. Moreover, the trajectories of massive objects and observers are always assumed to be future pointing.

We return to our particle, following a timelike, future pointing trajectory $\xi : I \to M$ in the Lorentzian manifold M. In analogy to Special Relativity we can use the proper time along ξ as a parameter:

Theorem 10.2 *A time-like curve ξ in a (time-oriented) Lorentzian manifold (M, g_{ab}) can always be parameterised by proper time, i.e. such that $g_{ab}(\xi(\tau))\dot{\xi}^a(\tau)\dot{\xi}^b(\tau) = -1$, without changing its time-orientation.*

The proof is analogous to that of Theorem 6.5. We may define the velocity and acceleration in terms of the proper time by

$$v^a := \dot{\xi}^a, \qquad a^a := \dot{\xi}^b \nabla_b \dot{\xi}^a.$$

Note that both expressions are independent of any choice of coordinates, whereas the expression $\ddot{\xi}^\mu(\tau)$ does depend on this choice. The problem is that, to quantify the change of v^a at different values of the proper time τ, we need to compare tangent vectors at different points of the spacetime M. To do this in a coordinate independent way, we need to use the covariant derivative. Also note that the norm of v^a is constantly 1, just like in Special Relativity. It follows that $v_a a^a = 0$.

If the rest mass of the particle is m_0, then its energy-momentum and the force are defined as the dual vectors

$$P_a := m_0 v_a, \qquad F_a := v^b \nabla_b P_a.$$

Here F_a includes all forces other than gravity. One may easily verify that if m_0 is constant, $F_a = 0$ is equivalent to the geodesic equation $v^a \nabla_a v^b = 0$.

Let us now consider an observer, who follows another future pointing, timelike trajectory $s \mapsto \alpha(s)$, which we also assume to be parameterised by proper time. The velocity of the observer is $\dot{\alpha}^a$. If α and ξ both go through the point $p \in M$, then the observer may measure properties of the particle. Choosing an inertial frame in T_pM and comparing with Special Relativity we define the energy E and the (spatial) momentum $\mathbf{P}_a$, as measured by the observer at p, to be

$$E = -P_a \dot{\alpha}^a = -m_0 \dot{\xi}_a \dot{\alpha}^a, \qquad \mathbf{P}_a = P_a - E\dot{\alpha}_a.$$

To obtain $\mathbf{P}_a$ we simply projected out the component along $\dot{\alpha}^a$.

The kinematical concepts we introduced above make use of the fact that for any $p \in M$, the tangent space $T_p(M)$ equipped with the pseudo-inner product $g_{ab}(p)$ is isomorphic to Minkowski space. We now want to discuss how this identification between T_pM and Minkowski space can be made even more precise.

Theorem 10.3 (Riemannian normal coordinates) *At any $p \in M$ there is a diffeomorphism $\exp_p$ of an open neighbourhood $W \subset T_pM$ containing 0, onto an open neighbourhood $U \subset M$ containing $p \in U$, with the following properties:*

1. *For any $v^a \in W$ and $\lambda \in [0,1]$, $\lambda v^a \in W$,*
2. *For any $v^a \in W$, the curve $\gamma : [0,1] \to M$ defined by $\gamma(\lambda) := \exp_p(\lambda v^a)$ is a geodesic with $\gamma(0) = p$ and $\dot{\gamma}^a(0) = v^a$.*

The proof uses results on how solutions of differential equations depend on their initial data. These can be used to show that the map $\exp_p$ is well-defined and C^∞. The fact that it is a diffeomorphism on some neighbourhood W of 0 follows from the inverse function theorem.

If we choose an orthonormal basis e_μ of T_pM, with e_0 timelike and with a dual basis $e^{*\mu}$, then we can construct a chart $\psi : U \to V$ by setting $\psi(x) := (e^{*1}(\exp_p(x)), \ldots, e^{*n}(\exp_p(x)))$. The coordinate system $x^\mu : e^{*\mu} \circ \exp_p$ has the following properties:

$$x^\mu(p) = 0, \qquad g_{\mu\nu}(p) = \eta_{\mu\nu}, \qquad \partial_{x^\rho} g_{\mu\nu}(p) = 0.$$

The last equality is equivalent to the vanishing of the Christoffel symbol at $p \in M$ in Riemannian normal coordinates, which follows from the geodesic equation (19) for the curves $\lambda \mapsto \lambda x^\mu$. This shows that the identification of T_pM with Minkowski space can even be made up to first order derivatives of the metric.

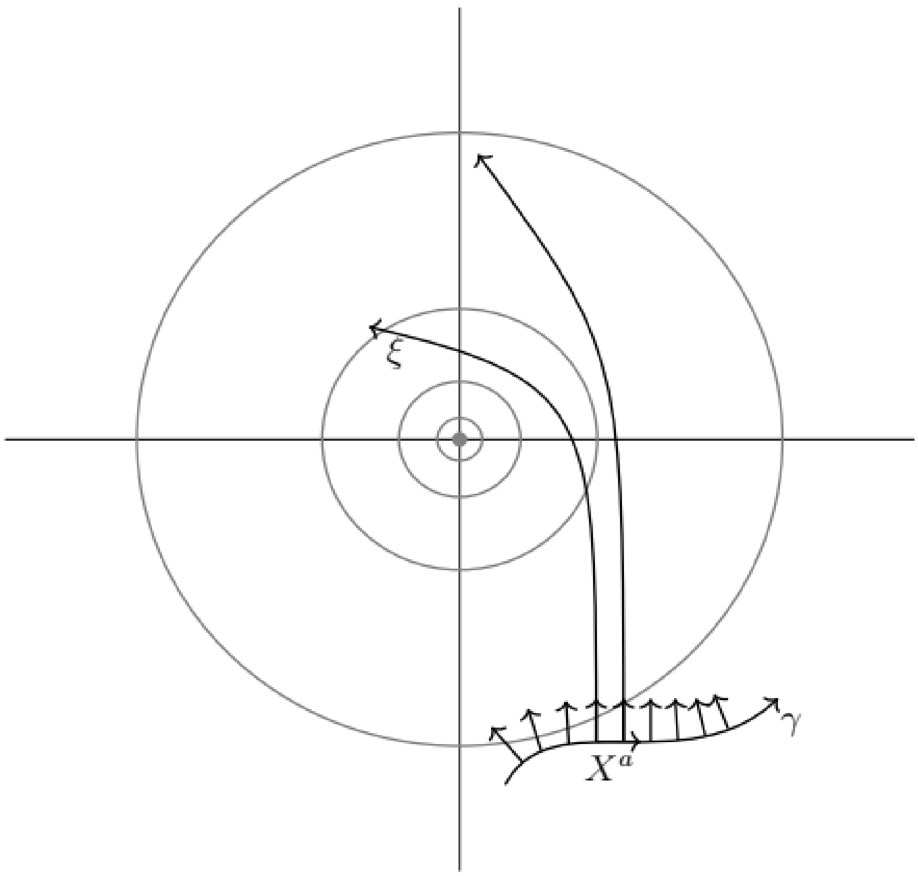
Note that the choice of coordinates heavily depends on the fact that $p \in M$ was fixed in advance, so in this coordinate system these special properties typically fail at any other point. However, it is possible to choose coordinates along any curve such that the metric takes the form $g_{\mu\nu} = \eta_{\mu\nu}$ and the derivatives $\partial_{x^\rho} g_{\mu\nu} = 0$ vanish all along the curve. (Such coordinates are called Fermi-Walker coordinates.) This explains how General Relativity satisfies the strong equivalence principle.

One cannot expect to do much better than Fermi-Walker coordinates, because the Riemann curvature tensor ${R_{abc}}^d$ is independent of the choice of coordinates, but its coordinate expression only depends on derivatives of the metric up to second order.

Definition 10.4 *A spacetime (M, g_{ab}) is called* flat*, when the Riemann curvature tensor vanishes, ${R_{abc}}^d = 0$.*

One can show that a flat spacetime can be covered by charts on which the metric takes the form $\eta_{\mu\nu}$ throughout the domain of the chart. I.e., we can cover M by open regions which look like open regions of Minkowski space (including the metric).

Our framework very nicely implements the strong equivalence principle, but how exactly does the metric g_{ab} encode gravity? The answer to this question is a bit subtle, because we like to think of gravity as a force which causes acceleration. In General Relativity, however, there is no longer any notion of absolute acceleration, because there is no preferred structure with respect to which something could be accelerated. (This is the ultimate consequence of treating all charts on an equal footing.) Nevertheless, we can investigate the relative acceleration of two freely falling objects, which will explain the relation between General Relativity and gravity.



Suppose that $\xi : (-a, a) \to M$ is a timelike, future pointing geodesic with $p := \xi(0)$ and let $X^a \in T_pM$ be a vector which is orthogonal to $\dot{\xi}^a$. We wish to displace $p = \xi(0)$ to an infinitesimally close point in the direction of X^a, track the displaced point as it follows its geodesic and then see how the displacement vector changes.

To make these ideas precise we construct a map $\gamma : (-a, a)^2 \to M$ as follows (for a suitably small $a > 0$): We define the geodesic $\gamma(s,0) := \exp_p(sX^a)$ and we extend $X^a \in T_pM$ to the vector field $X^a := \partial_s \gamma^a(s,0)$ along the curve. Note that X^a is parallely transported along the curve, because it is just the tangent vector field of a geodesic.

Next we let $T^a(\gamma(s,0))$ be the parallel transport of $\dot{\xi}^a(0) \in T_pM$ along the curve $\gamma(s,0)$. We then have $X^aT_a = 0$ along the curve $\gamma(s,0)$, because parallel transports preserve inner products. Finally we define $\gamma(s,t) := \exp_{\gamma(s,0)}(tT^a(\gamma(s,0)))$, i.e. we consider the geodesics generated by the vectors T^a on the curve $\gamma(s,0)$. We now extend T^a and X^a to the entire range of γ by setting $T^a := \partial_t \gamma^a(s,t)$ and $X^a := \partial_s \gamma^a(s,t)$. Note that T^a is a tangent to a geodesic $t \mapsto \gamma(s,t)$, so it is parallelly transported along this geodesic.

We have now constructed a one-parameter family of geodesics $t \mapsto \gamma(s,t)$ with $\gamma(0,t) = \xi(t)$. We may think of X^a on $\xi(t)$ as the relative position of, or the infinitesimal displacement to, a nearby geodesic. The relative velocity of the nearby geodesic is given by $V^b := T^a \nabla_a X^b$ and the relative acceleration by

$$A^c := T^a \nabla_a (T^b \nabla_b X^c).$$

Theorem 10.5 (Geodesic Deviation Equation) *In the notations above,*

$$A^a = -R_{bcd}{}^a T^b X^c T^d. \tag{21}$$

Proof: We first show that $T^a \nabla_a X^b = X^a \nabla_a T^b$. This follows from $\partial_t \partial_s = \partial_s \partial_t$ as follows. For any smooth function f on M we have $X(f) = \partial_s (f \circ \gamma)(s,t)$ and $T(f) = \partial_t (f \circ \gamma)(s,t)$. The right-hand sides can once again be interpreted as smooth functions on the range of γ, to which the vectors X and T can be applied. In this way we find $T(X(f)) = \partial_t \partial_s (f \circ \gamma) = X(T(f))$. Using the calculus of covariant derivatives, this means

$$\begin{aligned} 0 &= T^a \nabla_a (X^b \nabla_b f) - X^a \nabla_a (T^b \nabla_b f) \\ &= (T^a \nabla_a X^b) \nabla_b f - (X^a \nabla_a T^b) \nabla_b f + T^a X^b (\nabla_a \nabla_b f - \nabla_b \nabla_a f) \\ &= (T^a \nabla_a X^b) \nabla_b f - (X^a \nabla_a T^b) \nabla_b f. \end{aligned}$$

Since f was arbitrary, we must have $T^a \nabla_a X^b = X^a \nabla_a T^b$.

Incidentally, note that $T^a T_a = -1$ on the range of γ and hence, by the geodesic equation for T^a,

$$\begin{aligned} T^a \nabla_a (X^b T_b) &= T_b T^a \nabla_a (X^b) = T_b X^a \nabla_a (T^b) \\ &= \frac{1}{2} X^a \nabla_a (T_b T^b) = \frac{-1}{2} X^a \nabla_a 1 = 0. \end{aligned}$$

Since $X_b T^b = 0$ on $\gamma(s, 0)$, it follows that $X_b T^b = 0$ on the entire range of γ.

Now we use the geodesic equation for T^a and the definition of the Riemann curvature tensor to compute:

$$\begin{aligned} A^c &= T^a \nabla_a (T^b \nabla_b X^c) = T^a \nabla_a (X^b \nabla_b T^c) \\ &= (T^a \nabla_a X^b) \nabla_b T^c + X^b T^a \nabla_a \nabla_b T^c \\ &= (X^a \nabla_a T^b) \nabla_b T^c + X^b T^a \nabla_b \nabla_a T^c - X^b T^a R_{abd}{}^c T^d \\ &= X^a \nabla_a (T^b \nabla_b T^c) - R_{abd}{}^c T^a X^b T^d \\ &= -R_{abd}{}^c T^a X^b T^d. \end{aligned}$$

Relabelling indices yields the result. □

Equation (21) shows that nearby geodesics can exhibit relative acceleration, which is described by the Riemann curvature of the metric. Roughly speaking, the gravitational field corresponds to the curvature of the metric. This effect goes under the name *gravitational tidal forces*.

Dynamics of General Relativity

In Newton's theory of gravity, the strength of the gravitational force on a test-mass is determined by the matter in the universe. In General Relativity, gravity is encoded by the metric g_{ab}, so it seems reasonable that g_{ab} should satisfy an equation of motion, Einstein's equation, which also involves the matter in the universe. This equation is the topic of the present section.

Because General Relativity should reduce to Newton's Theory of Gravity in a suitable limit, let us first recall how the dynamics of the latter theory is formulated. For this we revert to classical spacetime, with a classical inertial coordinate system $(t, \mathbf{x})$, where t is an absolute time coordinate. Let us consider a continuous distribution of matter, described by a mass density ρ. This mass density gives rise to a gravitational potential Φ, defined by

$$\Phi(\mathbf{x}) := \int \frac{-G_N \rho(\mathbf{x}')}{\|\mathbf{x} - \mathbf{x}'\|} \mathrm{d}\,\mathbf{x}',$$

so that the acceleration that the mass distribution causes on any test-mass m is

$$\mathbf{g}(\mathbf{x}) = -\nabla \Phi(\mathbf{x}) = \int -G_N \rho(\mathbf{x}') \frac{\mathbf{x} - \mathbf{x}'}{\|\mathbf{x} - \mathbf{x}'\|^3} \mathrm{d}\,\mathbf{x}'.$$

These are continuous versions of the usual point mass formula for Newton's law of gravitation, where the test-mass m drops out, due to the (weak) equivalence principle. Note that ρ can be recovered from Poisson's equation:

$$\Delta\Phi(\mathbf{x}) = -\nabla \cdot \mathbf{g}(\mathbf{x}) = 4\pi G_N \rho(\mathbf{x}), \tag{22}$$

where all derivatives are in the spatial directions. This differential equation tells us how the gravitational potential Φ depends on the mass density distribution ρ. Einstein's equation should be analogous to Poisson's equation, but formulated in a relativistic context.

We may compare the acceleration of a test-mass in Newtonian gravity to the geodesics of General Relativity:

$$\ddot{\xi}^i(t) = -\nabla^i\Phi(\xi(t)) \quad \longleftrightarrow \ddot{\xi}^\mu(t) = -\Gamma^\mu{}_{\nu\rho}\dot{\xi}^\nu(t)\dot{\xi}^\rho(t),$$

where $i = 1,2,3$ and $\mu = 0,1,2,3$. Now suppose that we can choose coordinates x^μ such that x^0 is a time coordinate in which the variations of the metric are negligible, i.e. the gravitational field is constant. Also suppose that the velocity of the test-mass is small, so that $\dot{\xi}^\rho(t)$ is close to $(1,0,0,0)$. The comparison above then suggests that

$$\nabla^i\Phi \quad \longleftrightarrow \quad \Gamma^i{}_{00} = -\frac{1}{2}g^{i\mu}\partial_\mu g_{00}$$

and Φ might correspond to $\Phi \simeq -\frac{1}{2} - \frac{1}{2}g_{00}$. (The term $-\frac{1}{2}$ is needed to find the Minkowski metric component $g_{00} = -1$ in the absence of masses.)

To find a suitable correspondence for $\Delta\Phi$ we need to consider second order derivatives. For this purpose we will investigate the gravitational tidal forces from the point of view of Newton's Theory of Gravitation. Given $\Phi(\mathbf{x})$ we can consider two freely falling particles at time t, at positions $\mathbf{x}$ and $\mathbf{x}+\mathbf{h}$. Their respective accelerations will be $-\nabla\Phi(\mathbf{x})$ and $-\nabla\Phi(\mathbf{x}+\mathbf{h})$, so the relative acceleration is

$$-\nabla\Phi(\mathbf{x}+\mathbf{h}) + \nabla\Phi(\mathbf{x}).$$

To find the relative acceleration of a particle which is infinitesimally close to the particle at $\mathbf{x}$ we take the derivative with respect to $\mathbf{h}$ at $\mathbf{h} = 0$ in the direction $\mathbf{X}$, which yields

$$-(\mathbf{X} \cdot \nabla)\nabla\Phi(\mathbf{x}).$$

This is the gravitational tidal force in the setting of Newtonian gravity and should be compared with the result in General Relativity, Theorem 10.5:

$$-R_{bcd}{}^{a}T^{b}T^{d} \quad \longleftrightarrow \quad -\nabla_c \nabla^a \Phi.$$

Here the timelike vector T^b can be compared to the classical flow of absolute time and, due to the antisymmetry properties of the Riemann curvature tensor, the indices a and b only range over the spatial coordinates. Contracting over a and b yields a suitable analogue of the term $\Box\Phi$ in Poisson's equation:

$$R_{bd}T^{b}T^{d} \quad \longleftrightarrow \quad \Delta\Phi. \tag{23}$$

Now we turn to a relativistic formulation of the mass density ρ. Keeping in mind the lesson of Special Relativity that mass is energy, it makes sense to compare ρ with an energy density. For a continuous (or C^∞) distribution of matter in a spacetime M, we define the energy density at a point $p \in M$, from the perspective of an observer at $p \in M$, to be the quotient of the mass in a spatial region B divided by the volume of B, in the limit where B shrinks to the point p. To see what this limit entails we consider an example, consisting of a matter distribution in Minkowski space, which we assume to consist of particles which are all at rest in some inertial frame. Now consider two inertial observers, related by a Lorentz boost, but both going through the same point $p \in M$ with velocity vectors T^a and T'^b, respectively. Suppose that the first observer is also at rest, so he measures an energy density which consists entirely of the rest masses of the particles, leading to a mass density ρ, say. To compute the result for the second observer, we apply a Lorentz transformation. We recall the following two relevant effects: the rest masses get multiplied by a factor $\gamma = T_a T'^a$ and the lengths in the direction of motion get multiplied by a factor γ^{-1}. The energy density for the second observer therefore becomes $\rho' = \gamma^2 \rho$. The results so far can be formulated very neatly as follows:

$$\rho' = T_{ab}T'^{a}T'^{b}, \qquad T_{ab} := \rho T_a T_b.$$

Generalising this idea leads to the following:

Tenet 10.6 *A continuous matter distribution in a spacetime (M, g_{ab}) can be described by a symmetric tensor $T_{ab} = T_{ba}$, the* stress-energy-momentum tensor *(which we will often abbreviate to* stress tensor*, for convenience). For an observer with velocity $T^a \in T_pM$, the components of T_{ab} are interpreted as follows:*

1. *$T_{ab}T^aT^b$ is the energy density at p,*

2. *$T_{ab}v^aT^b$ is the momentum in the direction v^a, if $v_aT^a = 0$,*

3. *$T_{ab}v^aw^b$ is the (v^a, w^b)-component of the stress tensor, if $v_aT^a = 0$ and $w_aT^a = 0$.*

For normal matter, the stress tensor is conserved, *$\nabla^a T_{ab} = 0$, and it satisfies the* weak energy condition*: $T_{ab}T^aT^b \geq 0$ for all timelike vectors T^a.*

In Special Relativity, the conservation equation $\nabla^a T_{ab} = 0$ is equivalent to the conservation of energy-momentum. In General Relativity, this interpretation only holds when gravity does not exert forces on the matter.

Thus, given a timelike vector T^a and a stress tensor T_{ab} we choose the quantity $T_{ab}T^aT^b$ to correspond with the mass density ρ for the observer described by T^a. In order to obtaine the appropriate analogue of Poisson's equation in General Relativity, we note that we may also choose instead the correspondence

$$2\left(T_{ab} - \frac{1}{2}g_{ab}T^c{}_c\right)T^aT^b \quad \longleftrightarrow \quad \rho. \tag{24}$$

To see this, we note that the main contribution to $T^c{}_c$ comes from the energy density $T_{ab}T^aT^b$, because the stress components in the stress tensor are typically comparably small in units where $c = 1$. The values of the left-hand side is therefore very close to $T_{ab}T^aT^b$, so the correspondence is still physically reasonable.

Together with the correspondence in Equation (23) we may now rewrite Poisson's equation as $R_{ab}T^aT^b = 8\pi G_N \left(T_{ab} - \frac{1}{2}g_{ab}T^c{}_c\right)T^aT^b$. Assuming that this holds for all timelike vectors T^a leads to

$$R_{ab} = 8\pi G_N \left(T_{ab} - \frac{1}{2}g_{ab}T^c{}_c\right).$$

In order to obtain an equation which just contains T_{ab} on the right-hand side we proceed as follows. Contracting over the indices a and b we find $R = -8\pi G_N T^c{}_c$. This can be used to eliminate the term $T^c{}_c$ in favour of R, which can be brought to the other side of the equation. The result is *Einstein's equation*:

$$G_{ab} = R_{ab} - \frac{1}{2}g_{ab}R = 8\pi G_N T_{ab}. \tag{25}$$

Einstein's equation is a non-linear partial differential equation for the components of the metric g_{ab}, which occur in Einstein's tensor G_{ab} with derivatives up to second order. The metric also occurs in the equations of motion of any matter in the universe, which, in turn, determine the form of T_{ab}. One ought to solve all these equations of motion together.

From Bianchi's identity, $\nabla^a G_{ab} = 0$ (cf. Theorem 9.20) we immediately find $\nabla^a T_{ab} = 0$. This equation already imposes a strong, but physically justifiable restriction on the matter in the universe. Note that if we had chosen the correspondence $T_{ab}T^aT^b = \rho$ in our derivation of Einstein's equation, we would instead have found $R_{ab} = 8\pi G_N T_{ab}$ which would not have been consistent with the conservation of the stress tensor. In many cases, $\nabla^a T_{ab} = 0$ encodes the full equations of motion of this matter.

It is possible to formulate Einstein's equation as a Cauchy problem. This means that one can prescribe a manifold Σ of dimension 3 and initial values for the metric g_{ab} and its "normal" derivatives on Σ and then construct a unique, maximal manifold M with a Lorentzian metric g_{ab} which contains Σ as a spacelike hypersurface and whose metric has the correct data on Σ. We will not consider the rather technical aspects of this formulation.

Properties of General Relativity

With Einstein's equation in place, the content of General Relativity can be formulated concisely as follows (cf. [8]):

Tenet 10.7 (General Relativity) *Spacetime is a four-dimensional manifold with a Lorentzian metric, whose relation to the matter distribution in spacetime is given by Einstein's equation (25).*

Before we proceed to discuss a number of physical predictions and applications of this theory, let us pause to mention a few general properties of it.

If the Lorentzian manifold (M, g_{ab}) is a solution to Einstein's equation for some stress tensor T_{ab}, then we can immediately construct further solutions using embeddings:

Definition 10.8 *An* embedding *of manifolds M, M' of the same dimension, is a smooth injective map $\psi : M \to M'$ such that the range $\psi(M) \subset M'$ is open and the map $\psi^{-1} : \psi(M) \to M$ is also smooth.*

A diffeomorphism *is a surjective embedding* $\psi : M \to M$. *Because the inverse* $\psi^{-1} : M \to M$ *is also a diffeomorphism, the set of all diffeomorphisms of a manifold* M *forms a group.*

One particular kind of embedding is the *canonical inclusion map* $U \to M$ of a subset $U \subset M$. Another kind is the charts that were used to define manifolds in the first place.

Given any embedding $\psi : M \to M'$, we can use ψ and ψ^{-1} to map tangent vectors and tensors from a point $p \in M$ to a point $\psi(p) \in M'$ and back. Assuming that all laws of physics can be formulated in terms of tensor fields (or similar geometric objects), we can use ψ and ψ^{-1} to transport all fields from M' to M, just as we did for charts. Now suppose that M' has a Lorentzian metric g'_{ab} and various tensor fields collectively denoted by ϕ, and let use denote by $g_{ab} := \psi^* g'_{ab}$ respectively $\phi = \psi^* \phi'$ the metric and tensor fields that we obtain on M via the embedding ψ. If (g'_{ab}, ϕ') satisfies Einstein's equation on M', then so does (g_{ab}, ϕ) on M. In formulae, if we write Einstein's equation as $E_{ab}(g, \phi) = 0$ (E_{ab} being the left minus right side), then E_{ab} is such that

$$E_{ab}(g, \phi) = \psi^* E_{ab}(g', \phi')$$

for any embedding $\psi : M \to M'$ and (g_{ab}, ϕ) related to (g'_{ab}, ϕ) in the manner described above. The above property of the Einstein field equations is often referred to as *general covariance*. One can show that general covariance implies, under certain reasonable technical assumptions, that E_{ab} must be a local differential operator, i.e. $E_{ab}|_p$ can be written, at each p as a contraction of $(g_{ab}, g^{ab}, \phi, \dots, \nabla_{(a_1} \dots \nabla_{a_r)} \phi, R_{abcd}, \dots, \nabla_{(a_1} \dots \nabla_{a_s)} R_{abcd})|_p$, where the orders r, s are locally bounded (Peetre's theorem, Thomas replacement theorem). Of course, we want E_{ab} to be given by the Einstein field equation with a local matter stress tensor, so general covariance along does not imply Einstein's equation but would hold for a broad class of field equations.

Intuitively speaking, general covariance means that the laws of physics cannot contain any non-dynamical quantities that refer to spacetime other than the metric g_{ab} (and possibly the time-orientation and orientation of M), and any other dynamical fields ϕ on a manifold M, and that the laws are described by partial differential equations (Einstein field equation). An example of a non-dynamical quantity would be a preferred coordinate system, which we can think of as a collection of four scalar fields $x^0, \dots, x^3 : M \to \mathbb{R}$ such that $(\nabla_a x^0, \dots, \nabla_a x^3)|_p$ spans $T^*_p M$ at each point p. Of course, there is no difficulty having a *dynamical* coordinate system. In this case, our four scalar fields would have to enter the stress tensor T_{ab} in a generally covariant way. Such theories are sometimes called "Einstein-ether"-theories, because the four scalar fields locally define a frame. From a general perspective, there is nothing special about such theories compared to theories with other kinds of matter fields.

Another consequence of general covariance is that the individual points $p \in M$ have no intrinsic physical meaning. In fact, we can only identify a point p by giving it a physical description, e.g. that it is the point where certain physical fields (including the metric g_{ab}) take certain prescribed values. Now, if we apply a diffeomorphism ψ to the Lorentzian manifold (M, g_{ab}) to obtain the new field configurations $(M, \psi^* g'_{ab})$, then a point $p \in M$ remains fixed, but the field configurations do not, so the physical description will now determine the point $\psi(p)$, rather than p.

Apart from being generally covariant in the above sense, which, as we have seen, is a feature of a broad class of theories, Einstein's equation are also local in another, more subtle sense. This property has to do with the fact that the Einstein field equations are basically *hyperbolic* partial differential equations. The prototype of such a hyperbolic equation is the wave equation for a scalar field $\phi : M \to \mathbb{R}$ on a fixed spacetime (M, g_{ab}),

$$\nabla_a \nabla^a \phi = j \ .$$

Suppose (M, g_{ab}) does not have any grossly pathological causal features, in the sense that there is a space like embedded hyper surface $\Sigma \subset M$ with the property that every inextendible causal curve $\gamma : (a, b) \to M$ (meaning $\dot{\gamma}^a$ timeline or null everywhere) intersects Σ precisely once. Then it can be shown that the wave equation has a well-posed initial value problem: For each $j \in C_0^\infty(M)$, and each choice of $p, q \in C_0^\infty(\Sigma)$, there is a smooth solution ϕ having $\phi|_\Sigma = q, n^a \nabla_a \phi|_\Sigma = p$, where n^a is the unit normal to Σ. Furthermore, if we change j in the causal future of Σ, then ϕ does not change in the causal past of Σ. Likewise, if we change p, q inside some compact set $K \subset \Sigma$, then ϕ does not change outside the domain of causal influence of K, i.e. outside the set of points p that can be connected to K by a causal curve. Since there are many ways of choosing Σ, j, p, q, this implies a kind of locality (i.e. local dependence on the source and initial conditions). There is a sense in which Einstein's equations are also hyperbolic and possess an initial value formulation of the above kind, although this feature is complicated by the fact that Einstein's equations are also generally covariant. A detailed discussion is outside the scope of our lectures and can be found e.g. in [?]. Let us only mention here that this feature is tightly related to the special form of E_{ab} and is not shared by most other generally covariant field equations.