Stress-Tensor and Energy Conditions

Let us list some stress tensors for various matter models.

1. Incoherent matter (a.k.a. dust): pointlike particles that follow timelike paths without influencing each other. The flow of the particles can be described by a future pointing, timelike velocity vector field u^a with $u^a u_a = 1$ with $u^a \nabla_a u^b = 0$ (so each particle follows a geodesic). The stress tensor is

$$T_{ab} = \rho u_a u_b,$$

   where $\rho : M \to \mathbb{R}_{>0}$ is a positive mass density function that satisfies the continuity condition $\nabla_a(\rho u^a) = 0$.

2. Perfect fluids: analogous to dust, but now the particles exert a pressure P on each other, $P : M \to \mathbb{R}$. Using the same notations as before, the vector field u^a may no longer have geodesics as its integral curves. The stress tensor is

$$T_{ab} = \rho u_a u_b + P(u_a u_b + g_{ab}).$$

   Note that the weak energy condition is satisfied, because $\gamma := u_a \dot{\alpha}^a \geq 1$ and hence $(\rho + P)\gamma^2 - P \geq \rho \geq 0$. The conservation $\nabla^a T_{ab}$ does give rise to conditions on ρ, P and u^a, which may be interpreted in terms of fluid dynamics. (In the non-relativistic limit, they reduce to conservation of mass and Euler's equation).

3. Electromagnetic fields: In Special Relativity, an electromagnetic field is described by a one-form A_a, modulo a gauge equivalence relation, $A_a \sim 0$ iff $A_a = \nabla_a \lambda$ for some function λ. From A_a one constructs a field-strength tensor $F_{ab} := 2\nabla_{[a} A_{b]}$, which is antisymmetric. Its six independent components encode the electric and magnetic fields, which, however, are observer dependent. Let $e_\mu \in T_p M$ be an orthonormal frame and consider an observer at $p \in M$ with velocity vector e_0 (which

must be timelike and future pointing). Then components of the electric and magnetic fields can be identified as follows, for this observer:

$$F_{\mu\nu} = \begin{pmatrix} 0 & E_1 & E_2 & E_3 \\ -E_1 & 0 & B_3 & -B_2 \\ -E_2 & -B_3 & 0 & B_1 \\ -E_3 & B_2 & -B_1 & 0 \end{pmatrix}.$$

The energy density for the observer is $\frac{1}{2}(\|\mathbf{E}\|^2 + \|\mathbf{B}\|^2)$, which may be written in a coordinate independent way as $T_{ab}(e_0)^a(e_0)^b$ with

$$T_{ab} := F_a{}^c F_{cb} - \frac{1}{4} g_{ab} F^{cd} F_{cd}.$$

The fact that this stress tensor is conserved follows from Maxwell's equations in vacuum, which read

$$\nabla_{[a} F_{bc]} = 0, \qquad \nabla^a F_{ab} = 0.$$

4. A free scalar field: A free scalar field of mass $m \geq 0$ is a function $\phi : M \to \mathbb{R}$ satisfying the Klein-Gordon equation

$$\Box\phi + m^2\phi := \nabla^a \nabla_a \phi - m^2 \phi = 0.$$

It has a stress tensor given by

$$T_{ab} := \nabla_a \phi \nabla_b \phi - \frac{1}{2} g_{ab} (\nabla^c \phi \nabla_c \phi + m^2 \phi^2).$$

In the last two examples, the stress tensor is closely related to spacetime symmetries if any. We shall come back to this question later.

We have already mentioned that if γ is a timelike curve representing an observer with tangent $\dot{\gamma}^a = T^a$, then $\rho = T_{ab} T^a T^b$ has the interpretation of the energy density measured by this observer, whereas other components have the interpretation of the various pressures and stresses (see e.g. [2]). To discuss in general the relationship between pressures and energy density, it is sometimes useful to introduce an orthogonal tetrad adapted to γ. Such a tetrad by definition consists of 4 vectors $e^a_{(\mu)}, \mu = 0, 1, 2, 3$ such that $e^a_{(0)} = T^a$ and such that

$$g_{ab} e^a_{(\mu)} e^b_{(\nu)} = \eta_{\mu\nu}$$

where $\eta_{\mu\nu} = diag(-1,1,1,1)$ is the Minkowski metric. With this notation

$$\begin{aligned} T_{ab}e^a_{(0)}e^b_{(0)} = \rho &= \text{energy density} \\ T_{ab}e^a_{(0)}e^b_{(i)} = P_i &= \text{momentum density} \\ T_{ab}e^a_{(i)}e^b_{(j)} = \theta_{ij} &= \text{stresses} \end{aligned}$$

in the "orthogonal frame" defined by the tetrad. Different tetrads defining the same space- and time orientations are related by a proper orthochronous Lorentz transformation, $e'^a_{(\mu)} = \Lambda_\mu{}^\nu e^a_{(\nu)}$, and these change the energy density, pressure density, and stress density in the usual way familiar from Special Relativity.

Energy conditions specify in general terms properties expected to hold from physical considerations. These conditions are traditionally grouped into the following categories.

a. Weak energy condition (WEC): For all timelike or null T^a we have

$$T_{ab}T^aT^b \geq 0.$$

For a perfect fluid with stress tensor $T_{ab} = \rho u_a u_b + Ph_{ab}$ and fluid velocity u^a, this means that $\rho + P \geq 0$ and $\rho \geq 0$.

b. Null energy condition (NEC): We have

$$T_{ab}l^al^b \geq 0$$

for all null l^a. For a perfect fluid, it means $\rho + P \geq 0$. Note that the weak energy condition leads to the null energy condition.

c. Dominant energy condition (DEC): For all timelike T^a, the condition requires that the vector

$$f^b = -T^b{}_aT^a$$

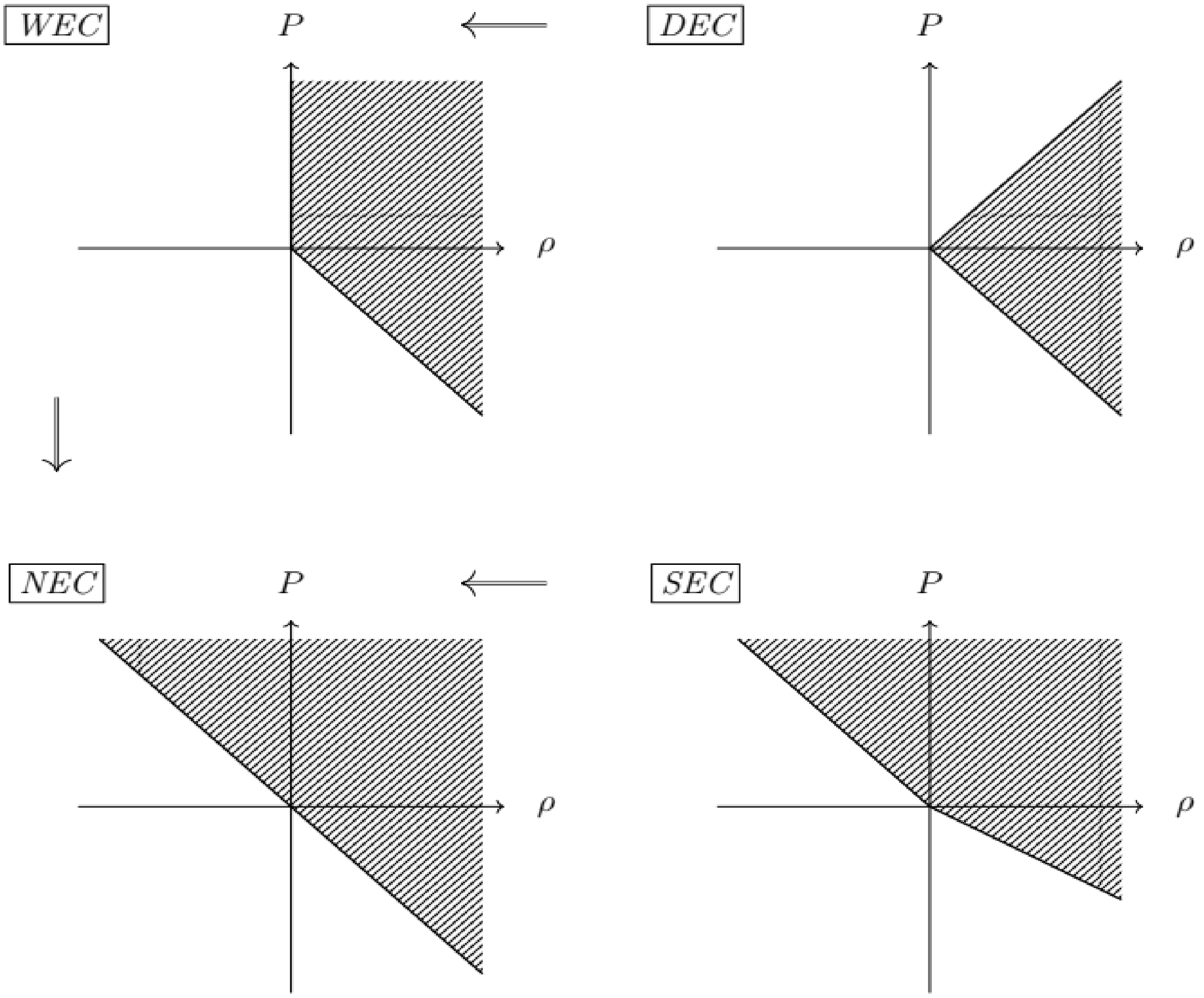
is a timelike or null future pointing vector. Note that the dominant energy condition leads to the weak energy condition. The dominant energy condition and Einstein's equation lead to the positive mass theorem. For a perfect fluid, it means that $\rho \geq |P|$.

d. Strong energy condition (SEC): For all timelike T^a, this condition requires that

$$T_{ab}T^aT^b \geq \frac{1}{2}T^c{}_cT^aT_a.$$

Note that the strong energy condition implies the null energy condition but it does not imply the weak energy condition. The strong energy condition and Einstein's equation leads to focussing of geodesics (see Raychaudhuri equation).

It is instructive to illustrate the meaning of the various energy conditions for a perfect fluid with stress tensor $T_{ab} = \rho u_a u_b + P h_{ab}$.



Part III

Applications of General Relativity

The conceptual changes and theoretical improvements of Einstein's General Relativity are by no means purely academic. Many subtle consequences of his theory have been experimentally verified during the past century and the theory now forms the basis of our understanding of the universe at large scales (cosmology), introducing the notion of a big bang and of black holes at the centres of galaxies. Closer to home, General Relativity is needed for the proper working of the Global Positioning System (GPS), because the older, less precise descriptions of radio signals in the Earth's gravitational field lead to errors of many meters. The most direct prediction of General Relativity, however, is that gravity is a field theory, so it propagates in the form of waves, rather than acting at a distance. A direct observation of these gravitational waves is difficult, because their effects are very small, but at present several experiments are underway and it is hoped that the first gravitational waves will soon be observed, leading to novel ways to look up into the sky and study the universe we live in.

Spacetime Symmetries

As in other branches of theoretical physics, in order to understand the implications of Einstein's equation, we have to study solutions describing real physical situations. Given the complexity of the equations, one must either

1. find solutions by numerical methods, or
2. make stringent symmetry assumptions and try to find analytic solutions.

The numerical analysis of Einstein's equations is a very important area of General Relativity, which is, however, beyond the scope of these notes. We therefore focus on the second option. It is surprising that solutions representing the two most important physical phenomena predicted by the theory can be found in this way:

1. cosmological solutions (Friedmann-Lemaître-Robertson-Walker metric) describing a dynamical cosmos,
2. black hole solutions (Schwarzschild, Kerr metric) describing objects from which no light can escape.

Before we come to these, it is useful and necessary to understand the nature of symmetries in General Relativity.

Since we describe spacetime geometry by a pair (M, g_{ab}) consisting of a manifold M and a Lorentzian metric g_{ab}, a symmetry is a map ψ preserving this structure. Thus, first of all ψ should be a diffeomorphism of M. In order to formulate that ψ preserves g_{ab}, we need the notions of pull-back and push-forward of tensors. The notion of push-forward is defined as follows. Let v^a be a vector at $p \in M$. Choose a curve $\gamma(s)$ in M whose derivative $\dot{\gamma}^a(0)$ equals v^a. The push forward of v^a, denoted $\psi_* v^a$, is the vector at $\psi(p)$ which is equal to the derivative $(d/dt)(\psi \circ \gamma)(0)$. The notion of pull-back applies to co-vectors and is defined by duality. Let w_a be a covector at $\psi(p)$. The pull back, denoted $\psi^* w_a$ is the covector at p whose action on v^a is related by

$$(\psi^* w_a) v^a \bigg|_p = w_a \psi_* v^a \bigg|_{\psi(p)} .$$

The notions of pull-back and push forward apply straightforwardly to tensors of higher rank. Coordinate expressions for the push-forward and pull-back are obtained as follows. Near p and $\psi(p)$ pick local coordinates x^μ and x'^μ. Let us write $\psi(x)^\mu = x'^\mu$ for the action of ψ in these local coordinates. (Note that there is slight abuse of notation here because ψ is really a map from M to M and not between coordinate vectors. What is meant here is the composition of ψ with the coordinate charts at p and $\psi(p)$). Then, if $t_{ab\dots c}$ is a tensor field with coordinate components $t_{\mu\nu\cdots\sigma}$, the pull-back has coordinate components

$$(\psi^\star t)_{\mu\nu\cdots\sigma}(x) = t_{\alpha\beta\dots\gamma}(x') \frac{\partial x'^\alpha}{\partial x^\mu} \cdots \frac{\partial x'^\gamma}{\partial x^\sigma} \tag{26}$$

where (x'^μ) is viewed as a function of (x^μ) in local coordinates (note the similarity with the transformation law for covariant tensors which of course is not accidental).

Definition 11.1 *A diffeomorphism ψ on M is called an isometry (or "symmetry") if $\psi^* g_{ab} = g_{ab}$.*

The isometries of a spacetime (M, g_{ab}) always form a Lie group because one can easily show that both the composition, as well as the inverse, of isometries are again isometries, and because these operations are continuous in a natural sense.

Example 11.2 (Minkowski space) *In the case* $M = \mathbb{R}^4, g_{ab} = \eta_{ab}$, *the isometries consist precisely of translations and Lorentz-transformations. The group of all isometries is isomorphic to* $O(3,1) \ltimes \mathbb{R}^4$, *the Poincare group.*

Just as the spacetime symmetries (isometries) form a Lie group, *infinitesimal symmetries* are associated with the Lie algebra of this group. Geometrically, infinitesimal symmetries are represented by vector fields ξ^a on M which, intuitively, describe "infinitesimal displacements". This can be made precise as follows. We call $\{\psi_t\}_{t\in\mathbb{R}}$ a 1-parameter group of diffeomorphisms if $\psi_t \circ \psi_s = \psi_{t+s}$ for all $s, t \in \mathbb{R}$ and if $\psi_{t=0} = id_M =$ identity on M. Given a 1-parameter group of diffeomorphisms, we can define a corresponding vector field ξ^a by the condition that

$$\frac{\partial}{\partial t} f(\psi_t(x))\Big|_{t=0} = \xi^a(x)\nabla_a f(x) \tag{27}$$

holds for all smooth real valued functions f on M. Given a covariant tensor field $t_{ab\dots c}$ on M, we call the operation

$$(\mathcal{L}_\xi t)_{ab\dots c} := \frac{\partial}{\partial t}(\psi_t^\star t)_{ab\dots c}\Big|_{t=0}$$

the *Lie-derivative*. The Lie derivative is independent of any choice of covariant derivative operator, but if we are given any covariant derivative operator, we may express the Lie derivative in terms of it:

$$(\mathcal{L}_\xi t)_{ab\dots c} = \xi^d \nabla_d t_{ab\cdots c} - (\nabla_a \xi^d) t_{db\cdots c} - (\nabla_b \xi^d) t_{ad\cdots c} \cdots - (\nabla_c \xi^d) t_{ab\cdots d}$$

In particular, we may choose $\nabla_a = \partial_a$ to be the flat derivative operator associated with some local coordinate system. This give a practical way to calculate the Lie derivative in coordinates, and to verify this formula using the coordinate expression for the pull-back, see (26).

Exercise 11.3 *Check that the right side is indeed independent of the choice of covariant derivative operator.*

Definition 11.4 *A vector field on M is called "Killing" for a spacetime metric g_{ab} on M if $(\mathcal{L}_\xi g)_{ab} = 0$. (Using the above expression for the Lie-derivative with ∇_a equal to the Levi Civita connection for g_{ab}, this is equivalent to*

$$\nabla_a \xi_b + \nabla_b \xi_a = 0 \quad (\textit{for } \xi_a = g_{ab}\xi^b)$$

since $\nabla_a g_{bc} = 0$.)

The relationship between Killing vectors and symmetries is the following. If $\{\psi_t\}_{t\in\mathbb{R}}$ is a 1-parameter group of symmetries of g_{ab}, then by definition, $\mathcal{L}_\xi g_{ab} = 0$. Conversely, if ξ^a is a Killing vector field, then there exist a 1-parameter group of symmetries $\{\psi_t\}_{t\in\mathbb{R}}$ such that (28) holds. Just as the isometries of a spacetime form a Lie group, the infinitesmial symmetries (Killing vector fields) form a Lie algebra whose commutator is given by the commutator of vector fields (see exercises).

Example 11.5 *For the symmetries in Example 11.2 we have:*

1. *For the translations in the z-direction, the associated infinitesmal generator (Killing vector field) is $(\xi^\mu) = (0,0,0,1)$ in inertial coordinates $(x^\mu) = (t,x,y,z)$.*
2. *For rotations around the z-axis, $(\xi^\mu) = (0,0,-z,y)$, and*
3. *for boosts along the x-axis, $(\xi^\mu) = (x,t,0,0)$.*

Cosmological Solutions to Einstein's Equation (Maximally Symmetric Universes)

Maximally symmetric universes are among the simplest, yet most important solutions to Einstein's equations. The form of their metric is restricted, up to one free function of time, by symmetry assumptions, whereas this function is determined by Einstein's equations. Let us first discuss the symmetries. We assume that $M = \mathbb{R} \times \Sigma$, and we assume that the line element is of the general form

$$ds^2 = -dt^2 + a(t)^2 \gamma_{ij}(x) dx^i dx^j$$

where $i,j = 1,2,3$ are coordinates on Σ. It is furthermore assumed that, for each fixed t, $\gamma_{ij} dx^i dx^j$ gives Σ the structure of a homogeneous, isotropic Riemannian manifold. Such a manifold by definition satisfies the following conditions:

1. Isotropy: For each $p \in \Sigma$ and any pair of unit vectors v^i, w^j at p there exists an isometry ψ of (Σ, γ_{ij}) such that $(\psi_\star v)^j = w^j$. For each orientation of Σ, there exists an isometry reversing the orientation.
2. Homogeneity: For each $p, p' \in \Sigma$ there exists an isometry ψ such that $\psi(p) = p'$.

The factor $a(t) > 0$ is called the *scale factor*. It is essentially the only non-trivial ingredient in the line element because it turns out that (Σ, γ_{ij}) must be maximally symmetric Riemannian spaces whose metric is basically fixed (locally) once we specify the (necessarily constant) curvature of (Σ, γ_{ij}). In order to see that (Σ, γ_{ij}) is basically "just symmetry", consider any fixed point $p \in \Sigma$, and let G be the isometry group. We may then define the "isotropy subgroup" G_0 leaving this point invariant. Any isometry $\psi \in G_0$ induces the linear isometric map $v^j \mapsto \psi_* v^j$ in the tangent space $T_p\Sigma$. Thus, G_0 is a subgroup of the group of linear isometric transformations of $T_p\Sigma$ (relative to the metric $\gamma_{ij}|_p$), and hence isomorphic to a subgroup of $O(n)$ (in n spatial dimensions). However, G_0 must actually be equal to $O(n)$. Otherwise, we could construct an invariant vector v^i (under G_0), or an invariant orientation. This is forbidden by our requirement of isotropy. If p' is any other point in Σ, there is an element $\psi \in G$ which carries p to p' by the homogeneity assumtion, and this element must be unique up to $\psi \mapsto \psi \circ \psi'$, where $\psi' \in G_0$. Hence, we can put points in Σ into one-to-one correspondence with elements of G modulo elements of $G_0 \cong O(n)$ (acting by right multiplication on G). In other words, Σ must be a quotient $G/O(n)$ of dimension n. It is then possible to see that the only possibilities for G can be $G = O(n+1), E(n), O(n,1)$, i.e. Σ is a maximally symmetric space. Furthermore, using this information, one can derive the local form of γ_{ij} using methods from Lie group theory.

Here, we will derive the local form γ_{ij} by a different, more explicit "by hand" method. One can show that a maximally symmetric space of dimension n necessarily has ([5]) a Riemann tensor of the form

$$\mathcal{R}_{ijkl} \;=\; k\,(\gamma_{ik}\gamma_{jl} - \gamma_{il}\gamma_{jk}),$$

where k in the last line is a real constant, and $\mathcal{R}_{ijkl}$ is the Riemann tensor of γ_{ij} (not to be confused with the $ijkl$-component of the Riemann tensor of the full spacetime metric!) [6]. In $n = 3$ spatial dimensions, the case of

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[6] As an aside, we note that this condition can also be stated as $D_i\mathcal{R}_{jklm} = 0$, where D_i is the Levi-Civita connection of γ_{ij}

interest for cosmology, possibile solutions to these equations will turn out to be locally isometric to

$$\begin{aligned}\Sigma &= \begin{cases}\mathbb{R}^3 & k=0,\ \text{3 dimensional flat space,}\\ \mathbb{H}^3 & k<0,\ \text{3 dimensional hyperbolic space,}\\ \mathbb{S}^3 & k>0,\ \text{3 dimensional sphere}\end{cases}\\ &\cong \begin{cases}{}^{E(3)}\!/_{O(3)}\\ {}^{O(3,1)}\!/_{O(3)}\\ {}^{O(4)}\!/_{O(3)}\end{cases} ,\end{aligned}$$

as we had already argued before. It is clear that our condition on the Riemann tensor (28) is preserved if we take a quotient of these spaces by a suitable finite subgroup of the isometry group, so we cannot expect to do better than finding the general solution up to quotients. However, as it turns out, these then give the general solution. We now derive these results by working out the consequences of our equation (28) for the Riemann tensor in $n = 3$ dimensions. In $n = 3$ spatial dimensions, the Ricci tensor and the Riemann tensor each have six independent components, so nothing is gained or lost by taking the trace of (28) (i.e. contraction with γ^{jl}). This gives

$$\mathcal{R}_{ij} = 2k\gamma_{ij}.$$

By the isotropy assumption, γ_{ij} must be spherically symmetric about each point p. It can be shown that this implies the existence of a local coordinate system $(x^i) = (r, \theta, \phi)$ such that p is at $r = 0$, such that (θ, ϕ) parameterize the orbits of $O(3)$ near p (each is $\cong S^2$ and (θ, ϕ) are spherical coordinates), and such that the line element takes the form

$$\gamma_{ij}dx^i dx^j = e^{2\beta(r)}dr^2 + r^2(d\theta^2 + \sin^2\theta\ d\phi^2) \tag{28}$$

for some function $\beta(r)$ which needs to be determined. The expression $r^2(d\theta^2 + \sin^2\theta d\phi^2)$ is of course nothing but the line element of the round 2-sphere of radius r. A calculation shows that the non-zero Ricci tensor components of this line element are:

$$\begin{aligned}\mathcal{R}_{11} &= \frac{2}{r}\beta'(r)\\ \mathcal{R}_{22} &= e^{-2\beta(r)}(r\beta'(r) - 1) + 1\\ \mathcal{R}_{33} &= \sin^2\theta\left\{e^{-2\beta(r)}\left(r\beta'(r) - 1\right) + 1\right\}\end{aligned}$$

whereas, by definition,

$$\begin{aligned} \gamma_{11} &= e^{2\beta(r)}, \\ \gamma_{22} &= r^2, \\ \gamma_{33} &= r^2 \sin^2\theta \ . \end{aligned}$$

So the 11-component of (28) gives:

$$\begin{aligned} \frac{2}{r}\beta' &= 2ke^{2\beta} \Longrightarrow \\ \frac{1}{2}e^{-2\beta} &= -\frac{1}{2}kr^2 + \frac{c}{2} \qquad (c = \text{ integration constant}) \quad \Longrightarrow \\ \beta &= -\frac{1}{2}\log\left(c - kr^2\right) . \end{aligned}$$

The 22-component of (28) gives:

$$\begin{aligned} 2kr^2 &= \underbrace{e^{-2\beta(r)}}_{(c-kr^2)} \underbrace{(r\beta'(r) - 1)}_{\left(\frac{kr^2}{c-kr^2} - 1\right)} + 1 \\ &= kr^2 - c + kr^2 + 1 \end{aligned} \tag{29}$$

from which it follows that $c = 1$. Consequently, the metric on Σ is locally given by:

$$\gamma_{ij}dx^i dx^j = \frac{dr^2}{1 - kr^2} + r^2(d\theta^2 + \sin^2\theta \ d\phi^2)$$

The final result can be interpreted as follows depending on the sign of k:

$\underline{k = 0}$: In this case $\mathcal{R}_{ijkl} = 0$. This should just be flat three dimensional Euclidean space. We can make this manifest by writing

$$\gamma_{ij}dx^i dx^j = dr^2 + r^2\left(d\theta^2 + \sin^2\theta \ d\phi^2\right) = dx^2 + dy^2 + dz^2$$

under the coordinate change (polar coordinates)

$$\begin{aligned} x &= r\sin\theta\cos\phi \\ y &= r\sin\theta\sin\phi \\ z &= r\cos\theta \end{aligned}$$

as one immediately verifies. The isometry group of flat Euclidean space is $E(3)$. The subgroup $O(3)$ is the isotropy group of any point, which makes manifest again that we can view this space as the quotient $E(3)/O(3)$.

$\underline{k>0}$: To get insight into the nature of this metric, we set $\sin^2\chi = kr^2$. Then we find

$$\gamma_{ij}dx^i dx^j = \frac{1}{k}\left(d\chi^2 + \sin^2\chi\left(d\theta^2 + \sin^2\theta d\phi^2\right)\right)$$

which is the round metric on $\mathbb{S}^3$ with the radius $\rho = 1/\sqrt{k}$, where (χ, θ, ϕ) are Euler angles. The isometry group of the round 3-dimensional sphere is $O(4)$. The isotropy group of any point is $O(3)$, which gives the representation of the three-sphere as $O(4)/O(3)$.

$\underline{k<0}$: This time, we set $\sinh^2\chi = -kr^2$. Then we find

$$\gamma_{ij}dx^i dx^j = \frac{1}{-k}\left(d\chi^2 + \sinh^2\chi\left(d\theta^2 + \sin^2\theta d\phi^2\right)\right).$$

This is seen to be the metric on a 3-dimensional hyperboloid $\mathbb{H}^3$ with "radius" $\rho = 1/\sqrt{-k}$. The terminology "hyperboloid" becomes clear if we present this space as the following subset of $\mathbb{R}^4$:

$$\mathbb{H}^3 = \left\{(u,x,y,z)| - u^2 + x^2 + y^2 + z^2 = \frac{1}{k}\right\},$$

equipped with the metric induced from 4-dimensional Minkowski spacetime. The correspondence with the previous parameterization is

$$\begin{aligned} x &= \rho\sinh\chi\sin\theta\cos\phi \\ y &= \rho\sinh\chi\sin\theta\sin\phi \\ z &= \rho\sinh\chi\cos\theta \\ u &= \rho\cosh\chi. \end{aligned}$$

The parameterization in terms u, x, y, z also makes manifest that the isometry group of the hyperboloid is $O(3,1)$. The isotropy subgroup of a point on the hyperboloid is $O(3)$, which gives the representation as the quotient $O(3,1)/O(3)$.

We note again that the metrics are only determined locally by the differential equations, so the general solution will be a quotient Σ/Γ of the spaces just found by suitable discrete subgroups Γ of the isometry groups. Examples are

$$\mathbb{T}^3 = \frac{\mathbb{R}^3}{\mathbb{Z}^3} \quad \text{or} \quad \frac{\mathbb{S}^3}{\mathbb{Z}_p} = \mathbb{L}(p,q) \tag{30}$$

The discrete subgroups are, in those cases, $\Gamma = \mathbb{Z}^3 \subset E(3)$ (flat case), $\Gamma = \mathbb{Z}_p \subset O(4)$ (positive curvature). The space $L(p,q)$ is called a Lens-space, and the natural number q which is co-prime to p is related to the precise definition of the action of $\mathbb{Z}_p$ on the 3-sphere: If we identify $\mathbb{R}^4$ with $\mathbb{C}^2$ and $\mathbb{S}^3$ with points $(z_1, z_2) \in \mathbb{C}^2$ having $|z_1|^2 + |z_2|^2 = 1$ then any $U(2)$ matrix can be mapped, via this identification, with an orthogonal matrix in $O(4)$ acting on the 3-sphere. If n is a natural number mod p, then its action is $(z_1, z_2) \mapsto (e^{in/p} z_1, e^{inq/p} z_2)$. The investigation of such quotients is an interesting mathematical problem. Physically, quotients do not seem to play a role in cosmology so far, so we will not discuss them further.

To summarize, our 4-dimensional spacetime manifold is

$$M = \mathbb{R} \times \begin{cases} \mathbb{R}^3 & k = 0, \text{ “flat universe”} \\ \mathbb{S}^3 & k > 0, \text{ “closed universe”} \\ \mathbb{H}^3 & k < 0, \text{ “open universe”} \end{cases}$$

and our 4-dimensional spacetime metric is

$$\begin{aligned} ds^2 &= -dt^2 + a(t)^2 \gamma_{ij}(x) dx^i dx^j \\ &= -dt^2 + a(t)^2 \left\{ \frac{dr^2}{1-kr^2} + r^2 (d\theta^2 + \sin^2\theta \; d\phi^2) \right\}. \end{aligned}$$

The spacetime metric depends, up to the parameter[7] $k \in \mathbb{R}$, only on the unknown function $a(t)$. This must be determined using Einstein's equation. For this, we begin by writing down the Ricci tensor components in the coordinates $(x^\mu) = (t, r, \theta, \phi)$. A calculation (excercises) shows:

$$\begin{aligned} R_{00} &= -3\frac{\ddot{a}}{a} \\ R_{11} &= \frac{a\ddot{a} + 2\dot{a}^2 + 2k}{1-kr^2} \\ R_{22} &= r^2(a\ddot{a} + 2\dot{a}^2 + 2k) \\ R_{33} &= r^2 \sin^2\theta (a\ddot{a} + 2\dot{a}^2 + 2k) \end{aligned}$$

To set up Einstein's equation, we must specify a stress tensor satisfying $\nabla^a T_{ab} = 0$ which is compatible with the spacetime symmetries, i.e. for all

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[7] To be precise, what matters is only the signum of k because we can always normalize k to $-1, 0$, or $+1$ by a change of coordinates $r \mapsto |k| r$ and a subsequent change of $a(t)$.

$\psi \in G$,

$$\psi^\star T_{ab} = T_{ab}.$$

It turns out that the most general such T_{ab} has the fluid form

$$T_{ab} = \rho u_a u_b + P(u_a u_b + g_{ab}) \ ,$$

where ρ, P are functions only of t and where $u_a = (dt)_a$ is the tangent field to the fluid lines. Equivalently, $(u^\mu) = (1,0,0,0)$ in our “comoving” (with the fluid) coordinates $(x^\mu) = (t, r, \theta, \phi)$, and

$$(T_\mu{}^\nu) = \begin{pmatrix} \rho & 0 & 0 & 0 \\ 0 & P & 0 & 0 \\ 0 & 0 & P & 0 \\ 0 & 0 & 0 & P \end{pmatrix} .$$

We complete our description of matter with an equation of state such as $P = w\rho$, where $w \in \mathbb{R}$ is constant. One usually requires that $|w| \leq 1$ (dominant energy condition) which includes the case of a “cosmological constant”, $T_{ab} = -\rho g_{ab}$ with $\rho = Lambda > 0$ the cosmological constant.

We can now solve Einstein's equation, which we will use in the trace reversed form $R_{ab} = 8\pi G_N \left(T_{ab} - {}^1\!/\!{}_2 g_{ab} T\right)$, where $T = g_{ab}T^{ab} = -\rho + 3P$ in our case. However, before investigating this equation directly, it is useful to look at the consequences of the equation $\nabla_a T^{ab} = 0$ (Bianchi-identity). This gives

$$0 = -\dot\rho - \frac{3\dot a}{a}\underbrace{(\rho + P)}_{(1+w)\rho}$$

which for $\rho > 0$ can be solved by

$$\begin{aligned} \frac{\dot\rho}{\rho} &= -3(1+w)\frac{\dot a}{a} \\ \Leftrightarrow \qquad \frac{d}{dt}\log\rho &= \frac{d}{dt}(\log a^{-3(1+w)}) \end{aligned} \tag{31}$$

Hence

$$\rho \propto a^{-3(1+w)} \ . \tag{32}$$

For some conventional forms of matter this gives

w	matter	
0	pressureless dust	$\rho \propto a^{-3}$
$\frac{1}{3}$	photons	$\rho \propto a^{-4}$
-1	cosmological constant	$\rho \propto a^{0}$

We now look at Einstein's equation directly. The 00-component gives

$$-3\frac{\ddot{a}}{a} = 4\pi G_N(\rho + 3P)$$

whereas any of the ii-components gives

$$\frac{\ddot{a}}{a} + 2\left(\frac{\dot{a}}{a}\right)^2 + \frac{2k}{a^2} = 4\pi G_N(\rho - P).$$

One can eliminate $\frac{\ddot{a}}{a}$ to get

$$\boxed{\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G_N}{3}\rho - \frac{k}{a^2}} \qquad \text{(first Friedmann equation)}$$

and

$$\boxed{\frac{\ddot{a}}{a} = -\frac{4\pi G_N}{3}(\rho + 3P)} \qquad \text{(second Friedmann equation)}$$

These two equations, first derived by Friedmann and independently by Lemaitre, encode the full dynamics of homogeneous, isotropic spacetimes. We point out the following very important

Qualitative features:

$\underline{k < 0}$: Then, $\dot{a}$ cannot become zero, which leads to a universe that is forever expanding.

$\underline{k = 0}$: For any matter having $P \geq 0$, ρ must decrease as a increases at least as rapidly as $\rho \approx a^{-3}$ (from the t-component of $\nabla_a T^{ab} = 0$). This causes ρa^2 to become zero. Thus, if $k = 0$, the expansion velocity $\dot{a}$ approaches zero as $t \to \infty$. (By contrast, for $k < 0$, $\dot{a} \to \sqrt{|k|}$ as $t \to \infty$.)

$\underline{k > 0}$: Then, the universe cannot expand forever, because the first term ρa^2 decreases faster than the k-term. There exists a critical $a = a_c$ so that $a \leq a_c$ for all time. Also, a cannot asymptotically approach a_c, as $-\ddot{a}$ is bounded below, which means that the universe 'bounces'.

We also note that, for any sign of k, the terms in the first Friedmann equation cannot balance each other for generic values of w. Thus, in view of (32), for any $w \geq -1$, $\dot{a}$ must diverge at early times, i.e. we are lead to the inevitable conclusion that under generic conditions, the universe must have started with a singular state having $\dot{a} = \infty$, i.e. a "big bang". This conclusion is one of the most important predictions of general relativity.

A table of some solutions is shown below. In this table, η is the "conformal" time coordinate, related to the time coordinate t by $a(\eta)d\eta = dt$. Furthermore, $C = \frac{1}{3}8\pi G_N \rho a^3$ and $C' = \frac{1}{3}8\pi G_N \rho a^4$ are constants

Geometry	Dust ($P = 0$)	Radiation ($P = \frac{\varrho}{3}$)
$k = +1 \quad (\mathbb{S}^3)$	$a = \frac{1}{2}C(1 - \cos\eta)$ $t = \frac{1}{2}C(\eta - \sin\eta)$	$a = \sqrt{C'}\sqrt{1 - \left(1 - \frac{t}{\sqrt{C'}}\right)^2}$
$k = 0 \quad (\mathbb{R}^3)$	$a = \sqrt[3]{\frac{9C}{4}}t^{\frac{2}{3}}$	$a = (4C')^{\frac{1}{4}}t^{\frac{1}{2}}$
$k = -1 \quad (\mathbb{H}^3)$	$a = \frac{1}{2}C(\cosh\eta - 1)$ $t = \frac{1}{2}C(\sinh\eta - \eta)$	$a = \sqrt{C'}\sqrt{\left(1 - \frac{t}{\sqrt{C'}}\right)^2 - 1}$

It is also useful to record that, for a flat universe ($k = 0$) and a general equation of state $P = w\rho$, with $w > -1$, we get (exercise)

$$a(t) \propto t^{\frac{2}{3(1+w)}}.$$

The limiting case $w = -1$ is also interesting and corresponds to a cosmological constant. In that case, an exponential expansion is found (exercise)

$$a(t) \propto e^{tH} \;, \tag{33}$$

where the constant $H = \dot{a}/a$ is called "Hubble constant".

One may object that the homogeous and isotropic spacetimes we have discussed could just be a mathematical exercise without much physical significance because our symmetry assumptions are clearly not satisfied in reality. Indeed, on small scales (e.g. molecular, solar system, or galactic), the stress energy is very far from being distributed in a homogeous and isotropic manner. Consequently, on such scales also spacetime metric is far from being homogeneous and isotropic (although less so, because the metric is "two derivatives down" from the stress tensor according to Poisson's/Einstein's equation). The hypothesis of homogeneity and isotropy seems, however, a very good approximation on cosmological scales, and the geometries we have described therefore lead to a good description of our universe in the very large. Current measurements indicate that $k = 0$ (or very nearly so) and that the scale factor appears to have been of the exponential type in the early cosmos (big bang, inflation), followed by a power law epoch (radiation dominated, then dust), and then followed again by an exponential type epoch. This behavior of the scale factor is incompatible with Einstein's equations if only conventional, baryonic, matter is assumed to be present. In particular, to get an exponential type expansion, we would need matter that is at least very similar to a "cosmological constant", as we have seen. The origin of such a hypothetical kind of matter component is still under discussion. At any rate, the prediction of a dynamical cosmos (for any type of matter) is one of the, if not the, most remarkable predictions of General Relativity.