Linearized Gravity and Gravitational Radiation

In the preceeding sections we have described (i) a dynamical solution to Einstein's equations with no "spatial excitations" (cosmological solutions) and (ii) a static solution with non-trivial spatial dependence (Schwarzschild black hole). A generic solution has spatial excitations that evolve in time. It turns out that the evolution equations implied by Einstein's equations have the character of a "quasi-linear" wave equation. The analysis of such equations is beyond the scope of these notes, see for example [?] for a detailed exposition. However, it turns out that one can derive, without major difficulties, the solutions describing small "linear" perturbations of Minkowski spacetime. These solutions describe gravitational waves. They have no counterpart in Newtonian Gravity and are, in this sense, a genuinely new prediction of General Relativity. This prediction is currently being tested with gravitational wave interferometers such as LIGO, GEO600 and VIRGO. It is conceivable that a detection might occur within this decade.

Gravitational waves in empty space

We first describe how to obtain the linearized Einstein equations. Mathematically, the best way to proceed is to suppose that one has a (differentiable) 1-parameter family $\{g_{ab}(s)\}_{s\in\mathbb{R}}$ of solutions to the Einstein equations, e.g. in vacuum,

$$R_{ab}(s) = 0$$

where we mean the Ricci tensor of the metric $g_{ab}(s)$. We think of $g_{ab} \equiv g_{ab}(0)$ as the "background", and we think of $g_{ab}(s), |s| \ll 1$ as small "deviations" or "perturbations" of this background. The first order deviation is just the derivative with respect to s at $s = 0$,

$$\gamma_{ab} \equiv \frac{\partial}{\partial s} g_{ab}(s) \bigg|_{s=0} .$$

γ_{ab} is referred to as the "linear perturbation". In General Relativity, the observable is not the metric, but its "gauge equivalence class", i.e. the set of all metrics related to g_{ab} by a diffeomorphism, $\psi^* g_{ab}$. Consequently, if $\{\psi(s)\}_{s\in\mathbb{R}}$ is a (differentiable) 1-parameter family of diffeomorphisms of M, then $\{g_{ab}(s)\}_{s\in\mathbb{R}}$ and $\{\psi(s)^* g_{ab}(s)\}_{s\in\mathbb{R}}$ should be viewed as physically describing the same families of spacetimes. Thus, at the linearized level (i.e. differentiating with respect to s and making use of the notion of Lie-derivatives), we find that γ_{ab} and $\gamma_{ab} + \mathcal{L}_X g_{ab}$ physically describe the same perturbation, where X^a is the generator of the family of diffeomorphisms at $s = 0$. One also says that the "gauge-invariance" at the linearized level is

$$\gamma_{ab} \to \gamma_{ab} + \mathcal{L}_X g_{ab} = \gamma_{ab} + \nabla_a X_b + \nabla_b X_a .$$

The linearized (vacuum) Einstein equations are obtained by simply differentiating the Einstein equations with respect to s at $s = 0$, namely

$$\dot{R}_{ab} \equiv \frac{\partial}{\partial s} R_{ab}(s) \bigg|_{s=0} = 0 .$$

The left side of this equation is readily calculated in terms of g_{ab} and γ_{ab}, but we will not give the general expression here. Rather we will specialize directly to the case of interest for us in which $g_{ab} = \eta_{ab}$ is Minkowski space. With $g_{ab}(s) = \eta_{ab} + s\gamma_{ab} + \mathcal{O}(s^2)$, we get:

$$\dot{g}^{ab} = -\gamma^{ab} \tag{49}$$

$$\dot{\Gamma}^c_{ab} = \frac{1}{2}\eta^{cd}\left(\partial_a \gamma_{bd} + \partial_b \gamma_{ad} - \partial_d \gamma_{ab}\right) , \tag{50}$$

where here and in the following, we adopt the convention that indices on expressions related to γ_{ab} are raised and lowered with η^{cd}, and where an overdot means a derivative with respect to s at $s = 0$. With these expressions at hand, we find that the linearized Riemann tensor is

$$\dot{R}_{eab}{}^{c} = -\frac{1}{2}\eta^{cd}\partial_e\left(\partial_a\gamma_{bd} + \partial_b\gamma_{ad} - \partial_d\gamma_{ab}\right) - (a \leftrightarrow e) \ ,$$

and we find that the linearized Ricci tensor is

$$\dot{R}_{ab} = \partial^c\partial_{(b}\gamma_{a)c} - \frac{1}{2}\partial^c\partial_c\gamma_{ab} - \frac{1}{2}\partial_a\partial_b\gamma \ ,$$

where $\gamma = \eta^{ab}\gamma_{ab}$ and where parenthesis denote symmetrization of the respective tensor indices. We then get for the linearized Einstein tensor:

$$\begin{aligned} \dot{G}_{ab} &= \dot{R}_{ab} - \frac{1}{2}\eta_{ab}\dot{R}_{cd}\eta^{cd} \\ &= \partial^c\partial_{(b}\gamma_{a)c} - \frac{1}{2}\partial^c\partial_c\gamma_{ab} - \frac{1}{2}\partial_a\partial_b\gamma - \frac{1}{2}\eta_{ab}\left(\partial^c\partial^d\gamma_{cd} - \partial^c\partial_c\gamma\right) \ . \end{aligned} \quad (51)$$

Using the formulae for Lie derivatives, the linearized gauge transformation on a Minkowski background is

$$\gamma_{ab} \to \gamma_{ab} + \partial_a X_b + \partial_b X_a \ ,$$

where X_a is an arbitrary (smooth) tensor. To investigate how these quantities change under a linearized gauge transformation We note that, for instance, $R_{abcd}[\psi^* g] = \psi^* R_{abcd}[g]$ for any metric g_{ef} and any diffeomorphism M (this of course expresses the "general covariance" of quantities like the Riemann tensor.) Using the formulas for the Lie-derivate, we consequently have for a linear perturbation of any background the transformation formula

$$\begin{aligned} \dot{R}_{abcd} &\to \dot{R}_{abcd} + \mathcal{L}_X R_{abcd} \ , \\ \dot{G}_{ab} &\to \dot{G}_{ab} + \mathcal{L}_X G_{ab} \ , \end{aligned} \quad (52)$$

and similarly for any other tensor field that is locally and constructed out of g_{ab}, ∇_a, g^{ab}. We can conclude from such formulae that any linearized quantity whose counterpart vanishes in the background, is automatically gauge invariant. In particular, if $G_{ab} = 0$ in the background, then the linearized Einstein tensor $\dot{G}_{ab}$ is gauge invariant. In Minkowski spacetime $R_{abcd} = 0$, so even the linearized Riemann tensor $\dot{R}_{abcd}$ is gauge invariant. We will now derive a wave equation for this quantity. For this, we look at the linearized Bianchi identities on Minkowski spacetime. They read

$$\begin{aligned} \partial_{[a}\dot{R}_{bc]de} &= 0 & (53) \\ \partial^{d}\dot{R}_{abcd} &= 0\ . & (54) \end{aligned}$$

We now apply ∂^a to the first equation and use the second equation as well as $\dot{R}_{ab} = 0$. Then we get, indeed,

$$\partial^a \partial_a \dot{R}_{bcde} = 0\ .$$

To analyze the effect of metric perturbations on the motion of test-observers, we choose the wave vector ωk_a, where

$$k^a = \frac{1}{\sqrt{2}}\left[\left(\frac{\partial}{\partial t}\right)^a - \left(\frac{\partial}{\partial z}\right)^a\right]$$

and consider a corresponding plane fronted wave-like perturbation γ_{ab} moving in the z-direction with spacetime dependence $\sin[\omega(t+z)]$. The corresponding linearized Riemann tensor is then a solution to the wave equation. To derive the motion of test-observers on this background, it is convenient to introduce another null vector l^a

$$l^a = \frac{1}{\sqrt{2}}\left[\left(\frac{\partial}{\partial t}\right)^a + \left(\frac{\partial}{\partial z}\right)^a\right]\ ,$$

and define the symmetric tensor

$$\Omega_{ab} \equiv \frac{1}{2}\ \dot{R}_{acbd} l^c l^d \qquad \Rightarrow \qquad \partial^c \partial_c \Omega_{ab} = 0\ .$$

The quantity Ω_{ab} has the following properties:

1. Ω_{ab} is invariant under linear gauge transformations.

2. $\Omega_{ab} k^b = 0$ (from the second linearized Bianchi identity and the dependence $\dot{R}_{abcd} \propto \sin[\omega(k^e x_e)]$), and $\Omega_{ab} l^b = 0$ (from $\dot{R}_{(ab)cd} = R_{ab(cd)} = 0$).

3. $\Omega_{ab}\eta^{ab} = 0$ (from $\dot{R}_{ab} = 0$), and $\Omega_{ab} = \Omega_{ba}$ (from $\dot{R}_{abcd} = \dot{R}_{cdab}$).

It follows that Ω_{ab} is a trace-free, symmetric tensor having components only in the x, y-directions. We can therefore write

$$\Omega_{ab} = \omega^2 \left\{ h_+ \cdot \epsilon^+_{ab} + h_\times \cdot \epsilon^\times_{ab} \right\} \sin[\omega(t+z)] \ ,$$

where $\{\epsilon^+_{ab}, \epsilon^\times_{ab}\}$ forms a basis of such tensors; in coordinates

$$\epsilon^+_{ab} = (dx)_a (dx)_b - (dy)_a (dy)_b \ , \qquad \epsilon^\times_{ab} = (dx)_a (dy)_b + (dx)_b (dy)_a \ ,$$

and where $h_+, h_\times \in \mathbb{R}$ are the amplitues of the "polarizations" $+, \times$. In inertial coordinates the polarization tensors for a wave moving in the z-direction are

$$(\epsilon^+_{\mu\nu}) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad (\epsilon^\times_{\mu\nu}) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \ .$$

We now consider the effect of a plane gravitational wave moving in the z-direction on the motion of test-observers. The motion of the test-observers is described by the geodesic deviation equation. We need to understand what this equation tells us at the order of approximation considered here. For this, it is best again to think about a family of metrics $g_{ab}(s) = \eta_{ab} + s\gamma_{ab} + \mathcal{O}(s^2)$. We consider geodesics starting on a slice $\Sigma = \{t = 0\}$ which are initially parallel with tangent vector $T^a = (\partial/\partial t)^a$. The evolution of these geodesics takes place in the s-dependent spacetime $g_{ab}(s)$. For $s = 0$, the metric is flat space and the geodesic deviation vector vanishes identically. Consequently, the geodesic deviation vector is Taylor expanded to *second* order in s as

$$X^a(s,t) = s\dot{X}^a(t) + \frac{1}{2}s^2\ddot{X}^a(t) + \mathcal{O}(s^3),$$

where an overdot again stands for a derivative with respect to the parameter s (and *not* the time parameter, t). We next differentiate the geodesic deviation equation several times with respect to s at $s = 0$. In principle, all quantities in the geodesic deviation equation depend upon the parameter s, including also $T^a = T^a(s,t)$, the tangent to the geodesics, because the metric $g_{ab}(s)$ depends upon s. Taking one s-derivative of the geodesic deviation equation gives

$$\frac{\partial^2}{\partial t^2}\dot{X}^a(t) = 0 \ ,$$

because the zeroth order deviation vector and Riemann tensor vanish. Since the geodesics are not initially diverging, it follows that $\dot{X}^a(t) = \dot{X}^a(0)$ is independent of t. Taking two s-derivatives gives

$$\begin{aligned}\frac{\partial^2}{\partial t^2}\ddot{X}^a(t) &= \dot{R}^a{}_{bcd}(t)T^b(0)T^d(0)\dot{X}^c(t) \ , &(55)\\ &= \Omega_{ab}(t)\dot{X}^a(0) \ , &(56)\end{aligned}$$

because at $s = 0$, $T^a(0,t) = (\partial/\partial t)^a \equiv T^a$ is constant, and because we have already seen that $\dot{X}^a(t) = \dot{X}^a(0)$ is constant, too. In the second line, we have also used the definition and properties of Ω_{ab}.

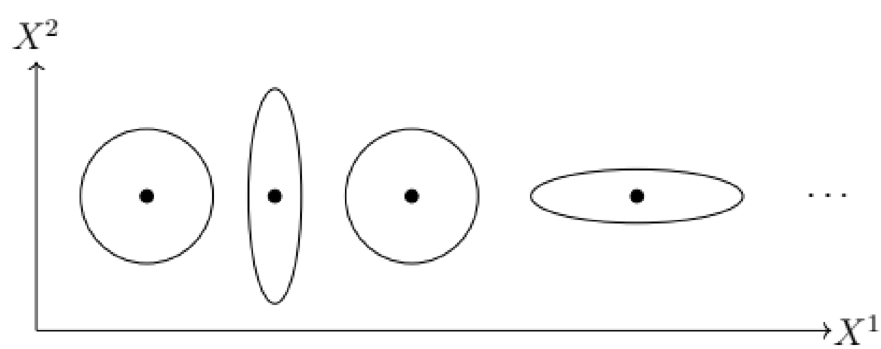
This equation may now be integrated. We can write $(X^\mu) = (0, X^1, X^2, 0)$ up to second order in the expansion in s, because the z-component must vanish, and the t-component vanishes by construction of the congruence. Integration gives for $h_\times = 0$

$$\left.\begin{aligned} X^1(t) &\approx X^1(0) + \tfrac{1}{2}h_+\sin(\omega t)\ X^1(0),\\ X^2(t) &\approx X^2(0) - \tfrac{1}{2}h_+\sin(\omega t))\ X^2(0).\end{aligned}\right\} \tag{57}$$

(Taylor expansion up to and including order s^2-terms with $s = 1$), whereas for $h_+ = 0$, we obtain

$$\left.\begin{aligned} X^1(t) &\approx X^1(0) + \tfrac{1}{2}h_\times\sin(\omega t)\ X^2(0),\\ X^2(t) &\approx X^2(0) + \tfrac{1}{2}h_\times\sin(\omega t)\ X^1(0).\end{aligned}\right\} \tag{58}$$

These displacements correspond to oscillations of a ring of test-masses (in the rest frame defined by T^a) in the (x, y)-plane as shown in the following figure.



We summarize our discussion as follows. At the linearized level, perturbations of Minkowski space (or rather, their corresponding Riemann tensor) obey a homogeneous wave equation. A plane wave moving in the z-direction gives rise to an oscillation of a ring of test-masses in the (x, y)-plane. The oscillation pattern depends on the polarization, $+$, respectively $\times$, and can potentially be observed e.g. by interferometers, as LIGO, VIRGO, GEO600 are indeed attempting to do. Since the physcial degrees of freedom of the gravitational field (i.e. gauge invariant information) can only manifest thenselves via their influence on test-masses, we can say that a gravitational wave has "two degrees of freedom" (per wave vector), namely $+, \times$.

Sources of gravitational waves

We next discuss the production of gravitational waves. For this, we need to study the linearized Einstein equations with a non-trival stress tensor T_{ab} representing the source. It is convenient at this stage to introduce the "trace reversed" variable

$$\overline{\gamma}_{ab} = \gamma_{ab} - \frac{1}{2}\eta_{ab}\gamma \qquad \Leftrightarrow \qquad \gamma_{ab} = \overline{\gamma}_{ab} - \frac{1}{2}\overline{\gamma}\eta_{ab} \ .$$

In terms of this variable, the linearized Einstein tensor takes the form

$$\begin{aligned} \dot{G}_{ab} &= -\frac{1}{2}\partial^c\partial_c\overline{\gamma}_{ab} + \partial^c\partial_{(a}\overline{\gamma}_{b)c} - \frac{1}{2}\eta_{ab}\partial^c\partial^d\overline{\gamma}_{cd} \\ &= 8\pi G_N\, T_{ab} \ . \end{aligned}$$

We have put a stress energy tensor on the right side. To be consistent at the linearized level, this should satisfy $\partial^a T_{ab} = 0$. Recall that the gauge invariance of General Relativity at the linearized level is

$$\gamma_{ab} \to \gamma_{ab} + \partial_a X_b + \partial_b X_a \ .$$

where the expression for the Lie derivative of the Minkowski metric has been used, and recall that $\dot{G}_{ab}$ is gauge invariant. We may use the gauge invariance to fix a particularly useful representer in the gauge equivalence class of γ_{ab}. For this, we note that under a gauge transformation

$$\partial^c\overline{\gamma}_{ac} \to \partial^c\overline{\gamma}_{ac} + \partial^c\partial_c X_a \ .$$

Because $\partial^c\partial_c$ is the wave operator in Minkowski spacetime, we can find a solution X_a to the equation $\partial^c\partial_c X_a = -\partial^c\overline{\gamma}_{ca}$. Using any X_a satisfying this equation in the gauge transformation, it follows that the gauge-transformed linear perturbation has $\partial^c \overline{\gamma}_{ac} = 0$. This gauge is called the "Lorentz gauge", by analogy with Maxwell's equations. In the Lorentz gauge, the linearized Einstein equation simply becomes

$$\Box \overline{\gamma}_{ab} = -16\pi G_N \, T_{ab}$$

We see again that the evolution equation for linear perturbations is a *wave equation*. It can be shown that the residual gauge freedom can be used up to impose even more stringent gauge conditions such as $\gamma = 0 = \gamma_{ab} T^b$ (in the source free region where $T_{ab} = 0$), see [Wald 1984] for a discussion. For a plane gravitational wave in empty space with wave-vector ωk_a as in the preceeding section, we thus get the conditions $\gamma = \gamma_{ab} T^b = \gamma_{ab} k^b = 0$. This reduces the number of independent components of γ_{ab} from 10 down to 2 (there are 8 independent gauge conditions). Thus, we see again that the gravitational field has 2 degrees of freedom per ωk_a, corresponding to $+, \times$ polarized waves. We next discuss the

Production of gravitational waves: Significant amounts of gravitational waves are produced in Nature by binary systems with large masses and large orbital frequencies, for instance by binaries of neutron stars or black holes (especially during their merger phase), but also in the Early Universe. Here we imagine a localized source such as a binary. The (in principle very complicated) structure of the source is supposed to be encoded in the matter stress tensor, T_{ab}. We imagine that T_{ab} has compact support in a spacetime region where for instance a collapse or merger takes place. From $\Box \overline{\gamma}_{ab} = -16\pi G_N \, T_{ab}$ we obtain

$$\overline{\gamma}_{ab} = -16\pi G_N \, \Delta^{\mathrm{ret}} * T_{ab} \; .$$

Here, the star is convolution, and Δ^{ret} is the retarded propagator

$$\Delta^{\mathrm{ret}}(t, \mathbf{x}) = -\frac{1}{2\pi} \Theta(-t) \delta(t^2 - \|\mathbf{x}\|^2)$$

From this, we get (as in electrodynamics) (with $G_N = 1$)

$$\begin{aligned} \overline{\gamma}_{ab}(t,\mathbf{x}) &= 8\int dt' d^3\vec{x}'\Theta(t-t')\delta\left((t-t')^2 - \|\mathbf{x}-\mathbf{x}'\|^2\right) T_{ab}(t',\mathbf{x}') \quad (59) \\ &= 4\int d^3x' \frac{T_{ab}(t',\mathbf{x}')}{\|\mathbf{x}-\mathbf{x}'\|}\Bigg|_{t-t'=\|\mathbf{x}-\mathbf{x}'\|} \\ &= 4\int\limits_{\dot{V}-(x)} \frac{T_{ab}(x')}{\|\mathbf{x}-\mathbf{x}'\|}\underbrace{ds(x')}_{=r^2 dr d\Omega} \end{aligned}$$

By taking a Fourier-transformation in t

$$\hat{\overline{\gamma}}_{ab}(\omega,\mathbf{x}) = \frac{1}{\sqrt{2\pi}}\int e^{i\omega t}\overline{\gamma}_{ab}(t,\mathbf{x})dt$$

we obtain from equation (59)

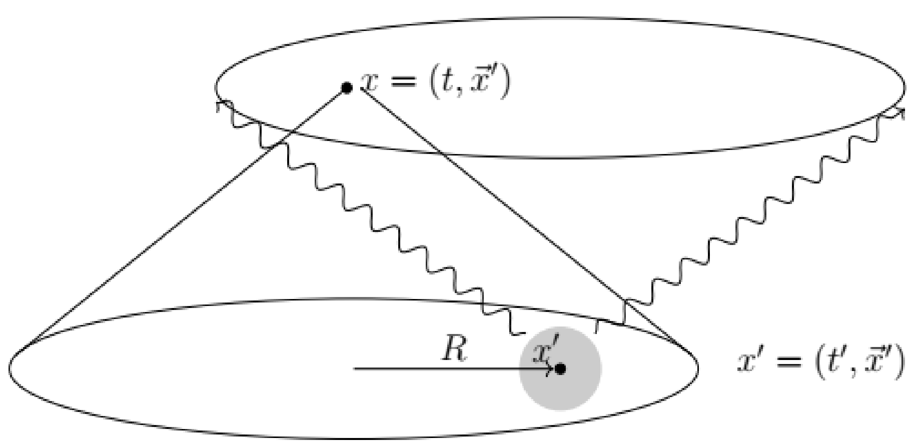
$$\hat{\overline{\gamma}}_{ab}(\omega,\mathbf{x}) = 4\int \frac{\hat{T}_{ab}(\omega,\mathbf{x}')}{\|\mathbf{x}-\mathbf{x}'\|}e^{i\omega\|\mathbf{x}-\mathbf{x}'\|}d^3\vec{x}'$$

The divergence-free condition $\partial^\mu\overline{\gamma}_{\mu\nu} = 0$ gives in Fourier-space

$$-i\omega\hat{\overline{\gamma}}_{0\nu} = \partial^j\hat{\overline{\gamma}}_{j\nu}$$

Now we assume that $R \gg \frac{1}{\omega}$ and $e^{i\omega\|\mathbf{x}-\mathbf{x}'\|}$ are nearly constant over the source, as seems physically reasonable (see the following figure). Then we can say that

$$\frac{e^{i\omega\|\mathbf{x}-\mathbf{x}'\|}}{\|\mathbf{x}-\mathbf{x}'\|} \approx \frac{e^{i\omega R}}{R}\ .$$



Now

$$\begin{aligned}
\int \hat{T}^{ij} d^3x &= \int \Big[\overbrace{\partial_k \left(\hat{T}^{kj} x^i \right)}^{=0 \text{ by Gauss theorem}} - \partial_k \hat{T}^{kj} x^i \Big] d^3x \\
&= -i\omega \int \hat{T}^{0j} x^i d^3x \\
&= -\frac{i\omega}{2} \int \left(\hat{T}^{0j} x^i + \hat{T}^{0i} x^j \right) d^3x \\
&= -\frac{i\omega}{2} \int \Big[\overbrace{\partial_k \left(\hat{T}^{0k} x^j x^i \right)}^{=0 \text{ by Gauss theorem}} - \partial_k \hat{T}^{0k} x^i x^j \Big] d^3x \\
&= -\frac{\omega^2}{2} \underbrace{\int \hat{T}^{00} x^i x^j d^3x}_{=\frac{1}{3}\hat{q}^{ij}(\omega) \text{ (quadrupole-tensor)}}
\end{aligned}$$

So we get

$$\hat{\bar{\gamma}}^{ij}(\omega, \mathbf{x}) \approx -\frac{2\omega^2}{3} \frac{e^{i\omega R}}{R} \hat{q}^{ij}(\omega)$$

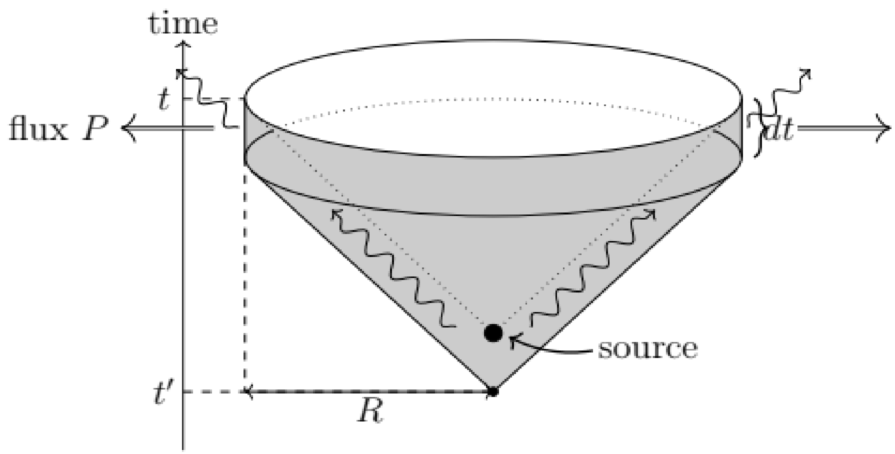
and after an inverse Fourier-transformation

$$\bar{\gamma}_{ij}(t, \vec{x}) \approx \frac{2}{3R} \frac{d^2}{dt^2} q_{ij} \underbrace{(t - R)}_{=\sqcup'}$$

where the approximation is valid in the slow motion and large-distance approximation.

One would like to calculate the energy flux of gravitational radiation, i.e. the energy emitted by the source per unit of time. In General Relativity, the notion of the total energy of a spacetime or parts thereof is actually not so easy to define, mainly due to the invariance of the theory under diffeomorphisms. We shall not discuss this complicated issue further, but note that, in the case of *linearized* gravity, a satisfactory notion of energy $E(t)$ associated with a suitable time slice $\Sigma(t)$ can be defined. With this notion, the flux, $P(t)$ is then defined as

$$\underbrace{P}_{\text{Flux}} dt = d \underbrace{E}_{\text{Energy}}$$



One way to define this energy E for linear perturbations is as follows: Set, for any pair of linearized perturbations:

$$w^a = \eta^{abcdef}\left(\gamma^{(1)}_{bc}\partial_d\gamma^{(2)}_{ef} - \gamma^{(2)}_{bc}\partial_d\gamma^{(1)}_{ef}\right)$$

where

$$\eta^{abcdef} = \eta^{ae}\eta^{fb}\eta^{cd} - \frac{1}{2}\eta^{ad}\eta^{be}\eta^{cf} - \frac{1}{2}\eta^{bc}\eta^{ae}\eta^{fd} - \frac{1}{2}\eta^{ab}\eta^{cd}\eta^{ef} + \frac{1}{2}\eta^{bc}\eta^{ad}\eta^{ef} \ .$$

A calculation using the linearized equations of motion $\dot{R}_{ab} = 0$ (for both perturbations) shows that $\partial^a w_a = 0$. Furthermore, define $j^a = w^a(\gamma, \partial_t\gamma)$. Then it can be shown that

1. $\partial_a j^a = 0$ in the source free region.
2. If $\Sigma(t)$ is a surface in the source free region, then

$$E(t) = \frac{1}{8\pi G_N}\int_{\Sigma(t)} j_a n^a dS$$

   is gauge invariant (here n^a is the unit normal to the surface and dS the integration element).

3. $E(t)$ is unchanged if we deform any compact subset of the surface $\Sigma(t)$.
4. $E(t)$ is decreasing with time in the source free region.

The proofs of these claims follow from the arguments in [7]. These properties suggest that $E(t)$ should be viewed as the energy of the linear perturbation γ_{ab} at "time t" (in the source free region) if we define $\Sigma(t)$ as a suitably "asymptotically hyperboloidal" slice approaching a lightcone at "retarded time", t. A possible choice is

$$\Sigma(t) = \{(x^\mu) \mid (x^0 - t)^2 - \sum_j (x^j)^2 = 1\} \ .$$

The corresponding flux $P(t)$ emitted by a gravitational wave in the far region may then be calculated. After a very lengthy calculation, it is found that

$$P(t) = \frac{d}{dt}E(t) \approx \frac{G_N}{45} \sum_{i,j=1}^{3} \left(\dddot{Q}_{ij}(t-R)\right)^2$$

where $Q_{ij} = q_{ij} - \frac{1}{3}\delta_{ij} q$ is the traceless part of the quadrupole tensor. This relation is known as the "quadrupole formula" and is to be compared with the corresponding formula in electromagnetism, involving only the dipole moment. This difference can be traced back to the difference in the tensor character of both fields.

To get an impression of the order of magnitude of gravitational radiation, we consider an

Example: We consider a binary consisting of point objects A and B with equal mass M, distance L in a circular orbit with orbital frequency Ω in the $z = 0$ plane. The stress tensor of the system is modeled by:

$$\begin{aligned} T^{00}(t,x,y,z) = \quad & M \quad \delta(z)\Big\{ \underbrace{\delta\left(x - L\cos\Omega t\right)\delta\left(y - L\sin\Omega t\right)}_{A} \\ & + \quad \underbrace{\delta\left(x + L\cos\Omega t\right)\delta\left(y + L\sin\Omega t\right)}_{B} \Big\} \end{aligned}$$

with all other components equal to 0. (Note that this form is actually inconsistent with $\partial^a T_{ab} = 0$, which is not surprising since it is not really possible to describe a bound system at the level of linear perturbation theory.) Proceeding despite this obvious inconsistency, we get for the reduced quadrupole (with $q_{ij} = 3\int T^{00} x_i x_j d^3x$)

$$
\begin{aligned}
Q_{ij} &= q_{ij} - \frac{1}{3}\delta_{ij} q \\
(Q_{ij}) &= \frac{ML^2}{3}\begin{pmatrix} 1+3\cos 2\Omega t & 3\sin 2\Omega t & 0 \\ 3\sin 2\Omega t & 1-3\cos 2\Omega t & 0 \\ 0 & 0 & -2 \end{pmatrix}
\end{aligned}
$$

Reinstating c, we get for the flux P of the binary:

$$P_{\text{binary}} = -\frac{128}{45c^5} G_N M^2 L^4 \Omega^6 \ .$$

The factor Ω^6 stems from 6 time derivatives and effectively turns P into an astronomically small number. For masses/lengths of the order of the Earth-Sun system, one finds a flux of about 100 Watts per second. To get an appreciable flux, one needs sources that are very massive and spinning very fast, and one needs sufficiently long observation times. Such sources are provided for example by pulsars, consisting of a pair of orbiting neutron stars. The orbital frequency Ω is observable to a high precision due to the "lighthouse effect". The gravitational radiation produced is sufficient to produce an energy loss affecting the orbital frequency. Such a change in the orbital frequency has now been observed in several pulsars. The quantitative results seem to confirm the quadrupole formula, and hence are a stringent test of General Relativity.

The Global Positioning System

The motivation behind the Global Positioning System is to accurately determine positions and times for any events near Earth. To do this, we need to take certain relativistic effects into account. The implementation of the GPS seems to be the first application of General Relativity on a large scale which is relevant to a general public because of commercial applications like car navigation equipment.

The results and presentation in section are mostly based on [1].

Introduction

Let us ignore gravity for the moment and suppose that we can describe all events near Earth using inertial coordinates $(ct, \mathbf{x})$ in the sense of Special

Relativity. Suppose that we (or an observer O) are at some unknown event $(ct_O, \mathbf{x}_O)$, but we receive radio signals from four sources and information about the four events $(ct_j, \mathbf{x}_j)$, $j = 1, 2, 3, 4$, where these signals originated. In general we can then determine the coordinates $(ct_O, \mathbf{x}_O)$ using a simple triangulation: because the radio signals travel at the speed of light, the four equations

$$t_O - t_j = c^{-1}\|\mathbf{x}_O - \mathbf{x}_j\|, \qquad j = 1, 2, 3, 4 \tag{60}$$

must be satisfied. These four equations allow us to determine the four unknown coordinates, in general.

A few remarks are in order:

1. If the signals are not received at exactly the same time, but at times $t_O + \delta_j$ and at positions $\mathbf{x}_O + \mathbf{d}_j$, we can use the equations

   $$t_O + \delta_j - t_j = c^{-1}\|\mathbf{x}_O + \mathbf{d}_j - \mathbf{x}_j\|, \qquad j = 1, 2, 3, 4.$$

   Note that the values for δ_j can be measured by the local observer (setting e.g. $\delta_1 = 0$ to eliminate a free constant). In addition we can make reasonable estimates for the vectors $\mathbf{d}_j$, because in most applications these are mostly due to the rotation of Earth. (The rotation speed depends on the latitude, but it is typically much larger than the speed of the observer with respect to Earth.) In this way we reduce the problem again to four unknowns, which may be obtained from four equations.

2. In the GPS, the sources are artificial satellites in orbit around Earth. There are 24 GPS satellites put into orbit in such a way that at least four of them are visible at almost every place and time on Earth. If the orbits of the satellites are known rather accurately as a function of time, it only remains for the satellite to determine the time, using an on-board clock.

3. In order to determine positions with an accuracy of 1m, we see from equation (60) that we need to determine times with an accuracy of $\frac{1\text{m}}{c\frac{\text{m}}{\text{s}}} \sim 3.33 \cdot 10^{-9}\text{s} = 3.33\text{ns}$.

High precision time measurement is one of the main challenges of the GPS system, and it is affected by the gravitational field of Earth and the motion of the satellites (and the observer O who wishes to determine its event). Some basic questions that are raised by the GPS and that involve relativity theory are:

1. How do we model the gravitational field of Earth?
2. What coordinates do we use to describe events?
3. How do we accurately measure the time coordinate using a clock on a satellite in orbit?
4. How do we accurately synchronise the clocks on various GPS satellites?
5. How do we communicate between satellites and the user, without losing accuracy?

The next subsections will address these issues.

Modelling the gravitational field of Earth

The gravitational field of Earth is rather complicated due to the details of its shape (not quite a sphere), its mass distribution (core vs. surface, e.g.) and its motion (rotation around an axis which itself is precessing, orbit around the sun). We will need to make some simplifying assumptions to deal with those issues, but if our assumptions are too strong, they will decrease the accuracy of our GPS.

We will neglect the dynamics of Earth's gravitational field, assume that the gravitational field is weak and that Earth has at least some symmetry, but not as much as spherical symmetry. In the weak field approximation to General Relativity, around a Minkowski background in spherical coordinates, we can write the metric as

$$ds^2 = \left(1 + \frac{2V}{c^2}\right) c^2 dt^2 - \left(1 - \frac{2V}{c^2}\right)(dr^2 + r^2 d\theta^2 + r^2 \sin^2(\theta) d\varphi^2), \quad (61)$$

where we can take V to be the Newtonian gravitational potential of Earth. This will be an approximate solution to Einstein's equation (it may be compared e.g. to the Schwarzschild metric).

If the mass distribution of Earth is given by a function $\rho(\mathbf{x})$, then the Newtonian gravitational potential is given by

$$V(\mathbf{x}) = -G_N \int \frac{\rho(\mathbf{y})}{\|\mathbf{x} - \mathbf{y}\|} d\mathbf{y}.$$

We take the coordinates to be centered on the centre of Earth and we use the Taylor expansion

$$\frac{1}{\|\mathbf{x}-\mathbf{y}\|} = \frac{1}{\|\mathbf{x}\|} + \sum_{i=1}^{3} y^i \frac{x^i}{\|\mathbf{x}\|^3} + \frac{1}{2}\sum_{i,j=1}^{3} y^i y^j \frac{3x^i x^j - \|\mathbf{x}\|^2\delta^{ij}}{\|\mathbf{x}\|^5} + \dots$$

to make a multi-pole expansion of V, namely

$$\begin{aligned} V(\mathbf{x}) &= \frac{-G_N M}{\|\mathbf{x}\|} - G_N \sum_{i=1}^{3} N^i \frac{x^i}{\|\mathbf{x}\|^3} - \frac{G_N}{2}\sum_{i,j=1}^{3} \frac{q^{ij}}{3}\frac{3x^i x^j - \|\mathbf{x}\|^2\delta^{ij}}{\|\mathbf{x}\|^5} + \dots \\ M &:= \int \rho(\mathbf{y})d\mathbf{y}, \\ N^i &:= \int y^i \rho(\mathbf{y})d\mathbf{y}, \\ q^{ij} &:= 3\int y^i y^j \rho(\mathbf{y})d\mathbf{y}. \end{aligned}$$

Here M is the total mass of the Earth. To find the other multi-pole coefficients we make some assumptions on the mass distribution ρ, namely that it has cylindrical symmetry around the x^3-axis, which we take to coincide with Earth's rotational axis, and that it has a reflection symmetry in the equatorial plane $x^3 = 0$. This means in particular that ρ is invariant if we change the sign of one of the coordinates y^i. It then easily follows that $N^i = 0$ and $q^{ij} = 0$ if $i \neq j$. Moreover, by the spherical symmetry, $q^{11} = q^{22}$. Using the fact that

$$\sum_{i=1}^{3} q^{11}\frac{3x^i x^i - \|\mathbf{x}\|^2\delta^{ii}}{\|\mathbf{x}\|^5} = 0$$

we then find

$$V(\mathbf{x}) \simeq \frac{-G_N M}{r}\left(1 - J_2\frac{a^2}{r^2}\frac{3\cos^2(\theta) - 1}{2}\right), \tag{62}$$

in spherical coordinates (r, θ, ϕ), where $a \sim 6.38 \cdot 10^6$m is Earth's radius at the equator and

$$J_2 = \frac{1}{3a^2 M}(q^{11} - q^{33}) \sim 1.08 \cdot 10^{-3}$$

is Earth's quadrupole moment coefficient. Higher multi-pole moments are not needed for GPS at the present level of accuracy, so our model for Earth's gravitational field consists of the metric (61) with V given by Equation (62).

Choice of coordinates

The form of the metric (61) already uses a set of coordinates (t, r, θ, ϕ) with a number of nice properties:

1. (t, r, θ, ϕ) are approximately inertial coordinates (written in spherical coordinates). They would be exactly inertial if we would ignore the gravitational field, setting $V = 0$. Because they are centered on the centre of Earth, they are almost inertial coordinates near Earth's geodesic world-line (compare to Fermi-Walker coordinates along a geodesic).

2. (r, θ, ϕ) nicely reflect the assumed symmetries: the rotation of Earth is described by a varying angular coordinate ϕ. Note that our coordinate system does not rotate along with Earth. (This would violate the approximately inertial property.)

3. t describes the proper time of a static observer at infinity, i.e. far away from Earth, where the potential V can be neglected altogether, because $V \sim 0$ when r is large.

These coordinates are very useful for describing e.g. the motions of the GPS satellites. However, they are less useful for the GPS users, who essentially rotate along with Earth. We therefore introduce an additional, rotating coordinate system:

$$t' = t, \qquad r' = r, \qquad \theta' = \theta, \qquad \phi' = \phi - \omega t, \tag{63}$$

where $\omega \sim 7.29 \cdot 10^{-5} \frac{\text{rad}}{\text{s}} \sim \frac{2\pi \text{ rad}}{24 \text{ hrs}}$ is the angular frequency of Earth's rotation. In these coordinates we have, up to order c^{-2}:

$$\begin{aligned} ds^2 &= \left(1 + \frac{2\Phi}{c^2}\right) c^2 dt'^2 - \frac{2\omega(r' \sin(\theta'))^2}{c} d\phi' \; cdt' \\ &\quad - \left(1 - \frac{2V}{c^2}\right)(dr'^2 + r'^2 d\theta'^2 + r'^2 \sin^2(\theta') d\varphi'^2), \end{aligned} \tag{64}$$

where $\Phi := V - \frac{(\omega r' \sin(\theta'))^2}{2}$ is an effective gravitational potential, which includes Earth's rotation as a centripetal potential term. Using Equation (62) for V we find

$$\Phi = \frac{-G_N M}{r'} + \frac{G_N M J_2 a^2}{r'^3} \frac{3\cos(\theta')^2 - 1}{2} - \frac{\omega^2 r'^2 \sin^2(\theta')}{2}. \tag{65}$$

To see how Earth's rotation influences time measurements we consider a clock at a fixed position on Earth, so r', θ', ϕ' are constant. The proper time τ is then related to the coordinate time t' by

$$d\tau = \left(1 + \frac{2\Phi}{c^2}\right)^{\frac{1}{2}} dt' \sim \left(1 + \frac{\Phi}{c^2}\right) dt', \tag{66}$$

up to order c^{-2}. At the equator we have $r' = a$, $\theta' = \frac{\pi}{2}$ and for $\Phi_0 := \Phi|_{\text{equator}}$ we have

$$c^{-2}\Phi_0 = \frac{-G_N M}{ac^2} - \frac{G_N M J_2}{2ac^2} - \frac{\omega^2 a^2}{2c^2} \sim -6.95 \cdot 10^{-10} - 3.76 \cdot 10^{-13} - 1.2 \cdot 10^{-12}.$$

The conclusion is that according to Equation (65), the (proper) time measured by a clock at the equator differs from the coordinate time t' by a change of rate or the order $\sim 7 \cdot 10^{-10}$.

Let us conclude this subsection with three remarks:

1. Within a matter of seconds, the error between $d\tau$ and dt' would reach our desired time accuracy $\sim 3 \cdot 10^{-9}$s. Note, however, that we ultimately want to compare time measurement on Earth with that on a satellite, not at infinity.

2. Two clocks which are fixed on Earth differ by a rate change that is determined by the values of Φ at their respective locations. These effects have to be taken into account when comparing clocks, e.g. for setting international time standards.

3. All clocks fixed on Earth at points where $\Phi = \Phi_0$ run at the same rate. These points form a surface, called the geoid of Earth.

4. For the time difference between $d\tau$ and dt' to add up to 1s, we need to wait $\sim 1.4 \cdot 10^9$s$\sim$ 44 years. One should expect the differences between the proper times ate various places on Earth to be smaller than that, so for practical purposes they are negligible. (There is an effect as in the twin-paradox when one person lives on the equator and the other on the North pole, say, but it is very small.)

Time measurement on a satellite

To compare time measurement on Earth with that on a satellite it is convenient to replace the time coordinate $t = t'$ by

$$t'' := \left(1 + \frac{\Phi_0}{c^2}\right) t' = \left(1 + \frac{\Phi_0}{c^2}\right) t,$$

which is the proper time measured by a clock fixed on Earth at the equator (or any other point on the geoid). The metric then takes the form

$$ds^2 = \left(1 + \frac{2(V - \Phi_0)}{c^2}\right) c^2 dt''^2 - \left(1 - \frac{2V}{c^2}\right)(dr^2 + r^2 d\theta^2 + r^2 \sin^2(\theta) d\varphi^2) \tag{67}$$

in non-rotating coordinates $(t'', r, \theta, \varphi)$.

Like any massive body, we can model the orbit of a satellite as a time-like curve $\gamma(t'') = (t'', r(t''), \theta(t''), \varphi(t''))$ with (coordinate dependent) velocity

$$v := \left(\dot{r}^2 + r^2\dot{\theta}^2 + r^2 \sin^2(\theta)\dot{\varphi}^2\right)^{\frac{1}{2}},$$

where $\dot{}$ denotes a derivative w.r.t. the time coordinate t''. The proper time along γ satisfies, up to order c^{-2},

$$\begin{aligned} c^2 d\tau^2 &= \left(1 + \frac{2(V - \Phi_0)}{c^2}\right) c^2 dt''^2 - \left(1 - \frac{2V}{c^2}\right) \frac{v^2}{c^2} c^2 dt''^2 \\ &\sim \left(1 + \frac{2(V - \Phi_0)}{c^2} - \frac{v^2}{c^2}\right) c^2 dt''^2 \\ d\tau &\sim \left(1 + \frac{V - \Phi_0}{c^2} - \frac{v^2}{2c^2}\right) dt''. \end{aligned}$$

This formula shows that the proper time along γ is affected by the gravitational field through V and by the motion through v. (The rate change of a clock due to its motion w.r.t. the coordinate system is known as the second order (relativistic) Doppler effect, because it depends in second order on $\frac{v}{c}$.)

Let us now describe the world-lines of the satellites in more detail, in order to find out the rates of their clocks in comparison to t''. The GPS satellites have orbits iwth an average altitude $r \sim 2.02 \cdot 10^7$m$\sim 3.2a$ (where a is again Earth's radius). At this altitude we can approximate $V \sim \frac{-G_N M}{r}$, because the quadrupole term falls of rapidly enough with the distance r. The satellite's motion is then accurately described by Newtonian gravity, leading to an elliptic orbit.

The distance r and the velocity v of the satellite change as it moves along its orbit. Because the elliptic orbits are simple enough, we can at least eliminate v from the problem as follows. We first note that the orbit takes place in a plane, where it can be described by the distance r and an angle ϕ, both depending on t''. Because Newtonian gravity is a conservative force, the total energy E (per unit mass of the satellite) is conserved, as is the angular momentum $L = r^2\dot{\phi}$. Adding kinetic and potential energy, E can be written as

$$E = \frac{v^2}{2} - \frac{G_N M}{r} = \frac{1}{2}\dot{r}^2 + \frac{L^2}{2r^2} - \frac{G_N M}{r}.$$

At the perigee (point farthest from Earth) and apogee (point closest to Earth), we have $\dot{r} = 0$ and $r = r_1$, respectively $r = r_0$, so that

$$E = \frac{L^2}{2r_1^2} - \frac{G_N M}{r_1} = \frac{L^2}{2r_0^2} - \frac{G_N M}{r_0}.$$

Eliminating L from these equations and using the fact that $r_0 + r + 1 = 2s$, where s is the semi-major axis of the elliptic orbit, we find

$$E = \frac{v^2}{2} - \frac{G_N M}{r} = -\frac{G_N M}{2s}.$$

This leads to

$$d\tau \sim \left(1 - \frac{\Phi_0}{c^2} - \frac{3G_N M}{2c^2 s} + \frac{2G_N M}{c^2}\left(\frac{1}{s} - \frac{1}{r}\right)\right) dt''.$$

Only the last term in brackets varies along the orbit, and it remains small when the orbit is close to being circular. The constant rate corrections

$$-\frac{\Phi_0}{c^2} - \frac{3G_N M}{2sc^2} \sim 6.97 \cdot 10^{-1} - 2.5 \cdot 10^{-10} \sim 4.46 \cdot 10^{-1}$$

can be implemented in the atomic clock before launch of the satellite (and after choosing the semi-major axis s).

Let us close this subsection with some remarks on the last result:

1. Equation (68) shows the change of rate between a clock on Earth and on a satellite, taking several relativistic effects into account, some of

which have opposite effects. One way of grouping these effects is as follows:

$$\frac{V - \Phi_0}{c^2} - \frac{v^2}{2c^2} = \frac{V - V_0}{c^2} + \frac{(\omega a)^2 - v^2}{2c^2},$$

where $V_o := V|_{\text{equator}}$. Here the first term on the right-hand side describes a gravitational blue-shift effect: the term is positive, indicating that clocks in orbit beat too fast. The second term describes the second order Doppler effects due to the motion of the satellite and the rotation of Earth. The satellites move faster than the rotating Earth (they circle Earth twice a day and their orbits are longer than the circumference of Earth), so this term is negative, indicating that clocks in orbit beat too slow.

2. Another way of grouping the various effects is as follows:

$$\begin{aligned} \frac{-3G_N M}{2c^2 s} - \frac{\Phi_0}{c^2} &= -\frac{G_N M}{c^2}\left(\frac{3}{2s} - \frac{1}{a}\right) + \frac{G_N M J_2}{2c^2 a} + \frac{(\omega a)^2}{2c^2} \\ &\sim 4.45 \cdot 10^{-10} + 3.76 \cdot 10^{-13} + 1.2 \cdot 10^{-12}, \end{aligned}$$

   where we have once again eliminated the velocity of the satellite in favour of its distance. Here the first term shows the effect of Earth's mass, which at our desired accuracy becomes relevant after ~ 7.5s. The second term shows the effect of Earth's shape, which becomes relevant after $\sim 8.9 \cdot 10^3$s$\sim$ 2.5hrs. The last term shows the effect of Earth's rotation, which becomes relevant after $\sim 2.8 \cdot 10^3$s$\sim$ 46min.

3. The variable rate correction $\frac{2G_N M}{c^2}\left(\frac{1}{s} - \frac{1}{r}\right)$ can add up to relevant contributions. It could be corrected by the software on the satellite before broadcasting the coordinates, but in the GPS this correction is left to the receiver.

How to synchronise clocks on different satellites.

Suppose that we have two GPS satellites in orbit, who measure proper times τ_1 and τ_2, respectively. So far we have only discussed how to adjust their clock rates to determine time differences in terms of t''. However, even after correcting the clock rate, there may still be a constant shift in the time coordinates determined by each satellite. To compensate for this shift we need to synchronise their clocks. Recall that the synchronisation of clocks which are located at different places is a non-trivial issue in relativity theory.

Let us suppose that satellite 1 measures the time t'', after correction of its clock rate. Then suppose that satellite 1 sends a signal at time t''_s, which arrives at time t''_a at satellite 2. According to the clock on satellite 2 the signal arrives at some time $\tilde{t}''_a$, which differs from t''_a by a constant, $\tilde{t}''_a - t''_a$. To synchronise the clock on satellite 2 with that on satellite 1 we need to adjust for this constant, i.e. we need to find out t''_a.

Because satellite 1 is a GPS satellite, it also broadcasts the coordinates of the event where the signal originates, so satellite 2 can act as a GPS receiver to find out the value of t''_s and the position coordinates $\mathbf{x}_s$ of the event when the signal was sent. In order to find t''_a we only need to know the distance l that the signal has travelled, because it travels at the constant speed of light c, so $l = c(t''_a - t''_s)$. When the signal arrives, satellite 2 is at position $\mathbf{x}_a$, so

$$l \sim \|\mathbf{x}_a - \mathbf{x}_s\|,$$

where we used the Minkowski metric as the lowest order in a weak field approximation to the spacetime metric. Higher order terms constribute corrections of the order c^{-1}. Thus,

$$t''_a = t''_s + c^{-1}\|\mathbf{x}_a - \mathbf{x}_s\|$$

up to order c^{-1}.

Communication with users on Earth

The satellites send out signals, which are electromagnetic waves of a certain frequency. For users on Earth it is often useful to measure this frequency, e.g. for determining velocities by the Doppler effect.

When expressed in the almost inertial coordinates (t'', r, θ, ϕ), the signal does not alter its frequency along its light-like geodesic from the satellite to the user, because the metric is static. However, we do need to take the relativistic Doppler effect into account which is caused by the motion of the satellite and the user w.r.t. the almost inertial coordinates.

The relativistic Doppler effect describes how the frequency of a signal changes under a change of (inertial) coordinate system. It can be expressed in terms of the velocity $\beta = \frac{v}{c}$ between the two coordinate systems. Making an expansion in terms of β we find no effect at order zero. The classical effect appears at first order and the relativistic effects occur at order two or higher. For our purposes we can distinguish:

1. A transversal effect:
   (The term transversal requires the choice of a coordinate system.) This effect is of second order β, so it is absent in the classical Doppler effect. In our case this effect has been accounted for already in the rate change of the satellite's clock (by the $-\frac{v^2}{2c^2}$-term).

2. A longitudinal effect:
   At first order in β this is the classical Doppler effect. Including its relativistic corrections, it states that the frequency changes according to

$$f = f' \sqrt{\frac{1-\beta}{1+\beta}} \simeq f'(1 - \beta + \ldots),$$

   where f and f' are the frequencies in the two relevant coordinate systems and β is the longitudinal component of $\frac{v}{c}$. This correction has to be applied for the change of coordinates at the satellite (from inertial coordinates in which the satellite is at rest, to the given coordinates) and at the user. The received frequency f_R is then related to the broadcast frequency f_0 by

$$f_R \simeq f_0 \frac{1 - c^{-1}\mathbf{N} \cdot \mathbf{v}_R}{1 - c^{-1}\mathbf{N} \cdot \mathbf{v}},$$

   where $\mathbf{N}$ denotes the direction of the signal's light-like geodesic, $\mathbf{v}_R$ is the velocity of the receiver and $\mathbf{v}$ that of the satellite.

These motion dependent effects are of order 10^{-5}, but they cannot be corrected in advance. Instead, by measuring the received frequency f_R, the user can reconstruct his velocity $\mathbf{v}_R$ using the formulae for the Doppler effect.

Conclusions

All in all, GPS consists of three so-called "segments":

1. Control segment:
   This consists of a number of monitoring stations, which gather information from the satellites, compute their orbits and (position dependent)

frequency corrections for the next few hours. This information is then uploaded to the satellites, to pass it on to the users. (Additional information which is monitored and passed on includes e.g. the "weather" in the ionosphere, which can affect the speed of light in that part of Earth's atmosphere significantly.)

2. Space segment:
   This consists of 24 satellites, carrying atomic clocks, with additional spare clocks and spare satellites. The satellites trasmit (a) timing signals, and (b) corresponding messages, specifying the coordinates of the timing signal's source event, as well as additional data needed to determine event coordinates.

3. User segment:
   This consists of all users which receive the satellite signals and use them to determine their position, time and velocity. Here we can distinguish two kinds of users: the commercial users and the U.S. military. Whereas the military receives the satellite signals on a restricted radio frequency, other users receive it on a frequency which can be received by commercially available receivers. However, the signal which is broadcast at the latter frequency is first distorted by small random noise, to reduce the accuracy. This is to prevent the use of GPS for unwanted military purposes by others than the U.S. military.