

Covariant derivative

Introduction

To formulate physical laws, we need to be able to differentiate tensor fields. For scalar fields, partial differentiation is fine: $f_{,\mu} \equiv \partial f / \partial x^\mu$ are the components of the covector field $(df)_a$. However, for tensor fields, partial differentiation is no good because the partial derivative of a tensor field does not give another tensor field:

Exercise. Let V^a be a vector field. In any coordinate chart, let $T^\mu{}_\nu = V^\mu{}_{,\nu} \equiv \partial V^\mu / \partial x^\nu$. Show that $T^\mu{}_\nu$ do *not* transform as tensor components under a change of chart.

The problem is that differentiation involves comparing a tensor at two infinitesimally nearby points of the manifold. But we have seen that this does not make sense: tensors at different points belong to different spaces. The mathematical structure that overcomes this difficulty is called a *covariant derivative* or *connection*.

Definition. A *covariant derivative* ∇ on a manifold M is a map sending every pair of smooth vector fields X, Y to a smooth vector field $\nabla_X Y$, with the following properties (where X, Y, Z are vector fields and f, g are functions)

$$\nabla_{fX+gY} Z = f\nabla_X Z + g\nabla_Y Z, \quad (4.1)$$

$$\nabla_X (Y + Z) = \nabla_X Y + \nabla_X Z, \quad (4.2)$$

$$\nabla_X (fY) = f\nabla_X Y + (\nabla_X f)Y, \quad (\text{Leibniz rule}), \quad (4.3)$$

where the action of ∇ on functions is defined by

$$\nabla_X f = X(f). \quad (4.4)$$

Remark. (4.1) implies that, at any point, the map $\nabla Y : X \mapsto \nabla_X Y$ is a linear

map from $T_p(M)$ to itself. Hence it defines a $(1,1)$ tensor (see examples sheet 1). More precisely, if $\eta \in T_p^*(M)$ and $X \in T_p(M)$ then we define $(\nabla Y)(\eta, X) \equiv \eta(\nabla_X Y)$.

Definition. let Y be a vector field. The *covariant derivative of Y* is the $(1,1)$ tensor field ∇Y . In abstract index notation we usually write $(\nabla Y)^a_b$ as $\nabla_b Y^a$ or $Y^a_{;b}$

Remarks.

1. Similarly we define $\nabla f : X \mapsto \nabla_X f = X(f)$. Hence $\nabla f = df$. We can write this as either $\nabla_a f$ or $f_{;a}$ or $\partial_a f$ or $f_{,a}$ (i.e. the covariant derivative reduces to the partial derivative when acting on a function).
2. Does the map $\nabla : X, Y \mapsto \nabla_X Y$ define a $(1,2)$ tensor field? No - equation (4.3) shows that this map is not linear in Y .

Example. Pick a coordinate chart on M . Let ∇ be the partial derivative in this chart. This satisfies all of the above conditions. This is not a very interesting example of a covariant derivative because it depends on choosing a particular chart: if we use a different chart then this covariant derivative will *not* be the partial derivative in the new chart.

Definition. In a basis $\{e_\mu\}$ the *connection components* $\Gamma^\mu_{\nu\rho}$ are defined by

$$\nabla_\rho e_\nu \equiv \nabla_{e_\rho} e_\nu = \Gamma^\mu_{\nu\rho} e_\mu \tag{4.5}$$

Example. The Christoffel symbols are the coordinate basis components of a certain connection, the *Levi-Civita connection*, which is defined on any manifold with a metric. More about this soon.

Write $X = X^\mu e_\mu$ and $Y = Y^\mu e_\mu$. Now

$$\begin{aligned} \nabla_X Y &= \nabla_X (Y^\mu e_\mu) = X(Y^\mu) e_\mu + Y^\mu \nabla_X e_\mu && \text{(Leibniz)} \\ &= X^\nu e_\nu (Y^\mu) e_\mu + Y^\mu \nabla_{X^\nu e_\nu} e_\mu \\ &= X^\nu e_\nu (Y^\mu) e_\mu + Y^\mu X^\nu \nabla_\nu e_\mu && \text{by (4.1)} \\ &= X^\nu e_\nu (Y^\mu) e_\mu + Y^\mu X^\nu \Gamma^\rho_{\mu\nu} e_\rho \\ &= X^\nu (e_\nu(Y^\mu) + \Gamma^\mu_{\rho\nu} Y^\rho) e_\mu \end{aligned} \tag{4.6}$$

and hence

$$(\nabla_X Y)^\mu = X^\nu e_\nu(Y^\mu) + \Gamma^\mu_{\rho\nu} Y^\rho X^\nu \tag{4.7}$$

so

$$Y^\mu_{;\nu} = e_\nu(Y^\mu) + \Gamma^\mu_{\rho\nu} Y^\rho \tag{4.8}$$

In a coordinate basis, this reduces to

$$Y^\mu{}_{;\nu} = Y^\mu{}_{,\nu} + \Gamma_{\rho\nu}^\mu Y^\rho \quad (4.9)$$

The connection components $\Gamma_{\nu\rho}^\mu$ are not tensor components:

Exercise (examples sheet 2). Consider a change of basis $e'_\mu = (A^{-1})^\nu{}_\mu e_\nu$. Show that

$$\Gamma'^\mu{}_{\nu\rho} = A^\mu{}_\tau (A^{-1})^\lambda{}_\nu (A^{-1})^\sigma{}_\rho \Gamma'_{\lambda\sigma}{}^\tau + A^\mu{}_\tau (A^{-1})^\sigma{}_\rho e_\sigma((A^{-1})^\tau{}_\nu) \quad (4.10)$$

The presence of the second term demonstrates that $\Gamma'^\mu{}_{\nu\rho}$ are not tensor components. Hence neither term in the RHS of equation (4.9) transforms as a tensor. However, the *sum* of these two terms does transform as a tensor.

Exercise. Let ∇ and $\tilde{\nabla}$ be two different connections on M . Show that $\nabla - \tilde{\nabla}$ is a $(1, 2)$ tensor field. You can do this either from the definition of a connection, or from the transformation law for the connection components.

The action of ∇ is extended to general tensor fields by the Leibniz property. If T is a tensor field of type (r, s) then ∇T is a tensor field of type $(r, s + 1)$. For example, if η is a covector field then, for any vector fields X and Y , we define

$$(\nabla_X \eta)(Y) \equiv \nabla_X(\eta(Y)) - \eta(\nabla_X Y). \quad (4.11)$$

It is not obvious that this defines a $(0, 2)$ tensor but we can see this as follows:

$$\begin{aligned} (\nabla_X \eta)(Y) &= \nabla_X(\eta_\mu Y^\mu) - \eta_\mu(\nabla_X Y)^\mu \\ &= X(\eta_\mu)Y^\mu + \eta_\mu X(Y^\mu) - \eta_\mu (X^\nu e_\nu(Y^\mu) + \Gamma_{\rho\nu}^\mu Y^\rho X^\nu), \end{aligned} \quad (4.12)$$

where we used (4.7). Now, the second and third terms cancel ($X = X^\nu e_\nu$) and hence (renaming dummy indices in the final term)

$$(\nabla_X \eta)(Y) = (X(\eta_\mu) - \Gamma_{\mu\nu}^\rho \eta_\rho X^\nu) Y^\mu, \quad (4.13)$$

which is linear in Y^μ so $\nabla_X \eta$ is a covector field with components

$$\begin{aligned} (\nabla_X \eta)_\mu &= X(\eta_\mu) - \Gamma_{\mu\nu}^\rho \eta_\rho X^\nu \\ &= X^\nu (e_\nu(\eta_\mu) - \Gamma_{\mu\nu}^\rho \eta_\rho) \end{aligned} \quad (4.14)$$

This is linear in X^ν and hence $\nabla \eta$ is a $(0, 2)$ tensor field with components

$$\eta_{\mu;\nu} = e_\nu(\eta_\mu) - \Gamma_{\mu\nu}^\rho \eta_\rho \quad (4.15)$$

In a coordinate basis, this is

$$\eta_{\mu;\nu} = \eta_{\mu,\nu} - \Gamma_{\mu\nu}^\rho \eta_\rho \quad (4.16)$$

Now the Leibniz rule can be used to obtain the formula for the coordinate basis components of ∇T where T is a (r, s) tensor:

$$\begin{aligned} T^{\mu_1 \dots \mu_r}_{\nu_1 \dots \nu_s; \rho} &= T^{\mu_1 \dots \mu_r}_{\nu_1 \dots \nu_s, \rho} + \Gamma^{\mu_1}_{\sigma \rho} T^{\sigma \mu_2 \dots \mu_r}_{\nu_1 \dots \nu_s} + \dots + \Gamma^{\mu_r}_{\sigma \rho} T^{\mu_1 \dots \mu_{r-1} \sigma}_{\nu_1 \dots \nu_s} \\ &- \Gamma^{\sigma}_{\nu_1 \rho} T^{\mu_1 \dots \mu_r}_{\sigma \nu_2 \dots \nu_s} - \dots - \Gamma^{\sigma}_{\nu_s \rho} T^{\mu_1 \dots \mu_r}_{\nu_1 \dots \nu_{s-1} \sigma} \end{aligned} \quad (4.17)$$

Exercise. Prove this result for a $(1, 1)$ tensor.

Remark. We are using a comma and semi-colon to denote partial, and covariant, derivatives respectively. If more than one index appears after a comma or semi-colon then the derivative is to be taken with respect to all indices. The index nearest to comma/semi-colon is the *first* derivative to be taken. For example, $f_{, \mu \nu} = f_{, \mu, \nu} \equiv \partial_\nu \partial_\mu f$, and $X^a_{; bc} = \nabla_c \nabla_b X^a$ (we cannot use abstract indices for the first example since it is not a tensor). The second partial derivatives of a function commute: $f_{, \mu \nu} = f_{, \nu \mu}$ but for a covariant derivative this is not true in general. Set $\eta = df$ in (4.16) to get, in a coordinate basis,

$$f_{; \mu \nu} = f_{, \mu \nu} - \Gamma^{\rho}_{\mu \nu} f_{, \rho} \quad (4.18)$$

Antisymmetrizing gives

$$f_{; [\mu \nu]} = -\Gamma^{\rho}_{[\mu \nu]} f_{, \rho} \quad (\text{coordinate basis}) \quad (4.19)$$

Definition. A connection ∇ is *torsion-free* if $\nabla_a \nabla_b f = \nabla_b \nabla_a f$ for any function f . From (4.19), this is equivalent to

$$\Gamma^{\rho}_{[\mu \nu]} = 0 \quad (\text{coordinate basis}) \quad (4.20)$$

Lemma. For a torsion-free connection, if X and Y are vector fields then

$$\nabla_X Y - \nabla_Y X = [X, Y] \quad (4.21)$$

Proof. Use a coordinate basis:

$$\begin{aligned} X^\nu Y^\mu_{; \nu} - Y^\nu X^\mu_{; \nu} &= X^\nu Y^\mu_{, \nu} + \Gamma^{\mu}_{\rho \nu} X^\nu Y^\rho - Y^\nu X^\mu_{, \nu} - \Gamma^{\mu}_{\rho \nu} Y^\nu X^\rho \\ &= [X, Y]^\mu + 2\Gamma^{\mu}_{[\rho \nu]} X^\nu Y^\rho \\ &= [X, Y]^\mu \end{aligned} \quad (4.22)$$

Hence the equation is true in a coordinate basis and therefore (as it is a tensor equation) it is true in any basis.

Remark. Even with zero torsion, the second covariant derivatives of a *tensor* field do *not* commute. More soon.

The Levi-Civita connection

On a manifold with a metric, the metric singles out a preferred connection:

Theorem. Let M be a manifold with a metric g . There exists a unique torsion-free connection ∇ such that the metric is covariantly constant: $\nabla g = 0$ (i.e. $g_{ab;c} = 0$). This is called the *Levi-Civita (or metric) connection*.

Proof. Let X, Y, Z be vector fields then

$$X(g(Y, Z)) = \nabla_X(g(Y, Z)) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z), \quad (4.23)$$

where we used the Leibniz rule and $\nabla_X g = 0$ in the second equality. Permuting X, Y, Z leads to two similar identities:

$$Y(g(Z, X)) = g(\nabla_Y Z, X) + g(Z, \nabla_Y X), \quad (4.24)$$

$$Z(g(X, Y)) = g(\nabla_Z X, Y) + g(X, \nabla_Z Y), \quad (4.25)$$

Add the first two of these equations and subtract the third to get (using the symmetry of the metric)

$$\begin{aligned} X(g(Y, Z)) + Y(g(Z, X)) - Z(g(X, Y)) &= g(\nabla_X Y + \nabla_Y X, Z) \\ &- g(\nabla_Z X - \nabla_X Z, Y) \\ &+ g(\nabla_Y Z - \nabla_Z Y, X) \end{aligned} \quad (4.26)$$

The torsion-free condition implies

$$\nabla_X Y - \nabla_Y X = [X, Y] \quad (4.27)$$

Using this and the same identity with X, Y, Z permuted gives

$$\begin{aligned} X(g(Y, Z)) + Y(g(Z, X)) - Z(g(X, Y)) &= 2g(\nabla_X Y, Z) - g([X, Y], Z) \\ &- g([Z, X], Y) + g([Y, Z], X) \end{aligned} \quad (4.28)$$

Hence

$$\begin{aligned} g(\nabla_X Y, Z) &= \frac{1}{2} [X(g(Y, Z)) + Y(g(Z, X)) - Z(g(X, Y)) \\ &+ g([X, Y], Z) + g([Z, X], Y) - g([Y, Z], X)] \end{aligned} \quad (4.29)$$

This determines $\nabla_X Y$ uniquely because the metric is non-degenerate. It remains to check that it satisfies the properties of a connection. For example:

$$g(\nabla_{fX} Y, Z) = \frac{1}{2} [fX(g(Y, Z)) + Y(fg(Z, X)) - Z(fg(X, Y))$$

$$\begin{aligned}
 & + g([fX, Y], Z) + g([Z, fX], Y) - fg([Y, Z], X)] \\
 & = \frac{1}{2} [fX(g(Y, Z)) + fY(g(Z, X)) + Y(f)g(Z, X) \\
 & - fZ(g(X, Y)) - Z(f)g(X, Y) + fg([X, Y], Z) - Y(f)g(X, Z) \\
 & + fg([Z, X], Y) + Z(f)g(X, Y) - fg([Y, Z], X)] \\
 & = \frac{f}{2} [X(g(Y, Z)) + Y(g(Z, X)) - Z(g(X, Y))] \\
 & + g([X, Y], Z) + g([Z, X], Y) - g([Y, Z], X)] \\
 & = fg(\nabla_X Y, Z) = g(f\nabla_X Y, Z)
 \end{aligned} \tag{4.30}$$

and hence $g(\nabla_{fX} Y - f\nabla_X Y, Z) = 0$ for any vector field Z so, by the non-degeneracy of the metric, $\nabla_{fX} Y = f\nabla_X Y$.

Exercise. Show that $\nabla_X Y$ as defined by (4.29) satisfies the other properties required of a connection.

Remark. In differential geometry, this theorem is called the *fundamental theorem of Riemannian geometry* (although it applies for a metric of any signature).

Let's determine the components of the Levi-Civita connection in a coordinate basis (for which $[e_\mu, e_\nu] = 0$):

$$g(\nabla_\rho e_\nu, e_\sigma) = \frac{1}{2} [e_\rho(g_{\nu\sigma}) + e_\nu(g_{\sigma\rho}) - e_\sigma(g_{\rho\nu})], \tag{4.31}$$

that is

$$g(\Gamma_{\nu\rho}^\tau e_\tau, e_\sigma) = \frac{1}{2} (g_{\sigma\nu,\rho} + g_{\sigma\rho,\nu} - g_{\nu\rho,\sigma}) \tag{4.32}$$

The LHS is just $\Gamma_{\nu\rho}^\tau g_{\tau\sigma}$. Hence if we multiply the whole equation by the inverse metric $g^{\mu\sigma}$ we obtain

$$\Gamma_{\nu\rho}^\mu = \frac{1}{2} g^{\mu\sigma} (g_{\sigma\nu,\rho} + g_{\sigma\rho,\nu} - g_{\nu\rho,\sigma}) \tag{4.33}$$

This is the same equation as we obtained earlier; we have now shown that the Christoffel symbols are the components of the Levi-Civita connection.

Remark. In GR, we take the connection to be the Levi-Civita connection. This is not as restrictive as it sounds: we saw above that the difference between two connections is a tensor field. Hence we can write any connection (even one with torsion) in terms of the Levi-Civita connection and a $(1, 2)$ tensor field. In GR we could regard the latter as a particular kind of "matter" field, rather than as part of the geometry of spacetime.

Geodesics

Previously we considered curves that extremize the proper time between two points of a spacetime, and showed that this gives the equation

$$\frac{d^2x^\mu}{d\tau^2} + \Gamma_{\nu\rho}^\mu(x(\tau)) \frac{dx^\nu}{d\tau} \frac{dx^\rho}{d\tau} = 0, \quad (4.34)$$

where τ is the proper time along the curve. The tangent vector to the curve has components $X^\mu = dx^\mu/d\tau$. This is defined only along the curve. However, we can extend X^μ (in an arbitrary way) to a neighbourhood of the curve, so that X^μ becomes a vector field, and the curve is an integral curve of this vector field. The chain rule gives

$$\frac{d^2x^\mu}{d\tau^2} = \frac{dX^\mu(x(\tau))}{d\tau} = \frac{dx^\nu}{d\tau} \frac{\partial X^\mu}{\partial x^\nu} = X^\nu X^\mu{}_{,\nu}. \quad (4.35)$$

Note that the LHS is independent of how we extend X^μ hence so must be the RHS. We can now write (4.34) as

$$X^\nu (X^\mu{}_{,\nu} + \Gamma_{\nu\rho}^\mu X^\rho) = 0 \quad (4.36)$$

which is the same as

$$X^\nu X^\mu{}_{;\nu} = 0, \quad \text{or} \quad \nabla_X X = 0. \quad (4.37)$$

where we are using the Levi-Civita connection. We now extend this to an arbitrary connection:

Definition. Let M be a manifold with a connection ∇ . An *affinely parameterized geodesic* is an integral curve of a vector field X satisfying $\nabla_X X = 0$.

Remarks.

1. What do we mean by "affinely parameterized"? Consider a curve with parameter t whose tangent X satisfies the above definition. Let u be some other parameter for the curve, so $t = t(u)$ and $dt/du > 0$. Then the tangent vector becomes $Y = hX$ where $h = dt/du$. Hence

$$\nabla_Y Y = \nabla_{hX}(hX) = h\nabla_X(hX) = h^2\nabla_X X + X(h)hX = fY, \quad (4.38)$$

where $f = X(h) = dh/dt$. Hence $\nabla_Y Y = fY$ describes the same geodesic. In this case, the geodesic is *not* affinely parameterized.

It always is possible to find an affine parameter so there is no loss of generality in restricting to affinely parameterized geodesics. Note that the new parameter also is affine iff $X(h) = 0$, i.e., h is constant. Then $u = at + b$ where a and b are constants with $a > 0$ ($a = h^{-1}$). Hence there is a 2-parameter family of affine parameters for any geodesic.

2. Reversing the above steps shows that, in a coordinate chart, for any connection, the geodesic equation can be written as (4.34) with τ an arbitrary affine parameter.
3. In GR, curves of extremal proper time are timelike geodesics (with ∇ the Levi-Civita connection). But one can also consider geodesics which are not timelike. These satisfy (4.34) with τ an affine parameter. The easiest way to obtain this equation is to use the Lagrangian (3.26).

Theorem. Let M be a manifold with a connection ∇ . Let $p \in M$ and $X_p \in T_p(M)$. Then there exists a unique affinely parameterized geodesic through p with tangent vector X_p at p .

Proof. Choose a coordinate chart x^μ in a neighbourhood of p . Consider a curve parameterized by τ . It has tangent vector with components $X^\mu = dx^\mu/d\tau$. The geodesic equation is (4.34). We want the curve to satisfy the initial conditions

$$x^\mu(0) = x_p^\mu, \quad \left(\frac{dx^\mu}{d\tau} \right)_{\tau=0} = X_p^\mu. \quad (4.39)$$

This is a coupled system of n ordinary differential equations for the n functions $x^\mu(t)$. Existence and uniqueness is guaranteed by the standard theory of ordinary differential equations.

Exercise. Let X be tangent to an affinely parameterized geodesic of the Levi-Civita connection. Show that $\nabla_X(g(X, X)) = 0$ and hence $g(X, X)$ is constant along the geodesic. Therefore the tangent vector cannot change e.g. from timelike to null along the geodesic: a geodesic is either timelike, spacelike or null.

Postulate. In GR, free particles move on geodesics (of the Levi-Civita connection). These are timelike for massive particles, and null for massless particles (e.g. photons).

Remark. In the timelike case we can use proper time as an affine parameter. This imposes the additional restriction $g(X, X) = -1$. If τ and τ' both are proper times along a geodesic then $\tau' = \tau + b$ (i.e. $a = 1$ above). In other words, clocks measuring proper time differ only by their choice of zero. In particular, they measure equal time intervals. Similarly in the spacelike case (or on a Riemannian manifold), we use arc length s as affine parameter, which gives $g(X, X) = 1$ and $s' = s + b$. In the null case, there is no analogue of proper time or arc length and so there is a 2-parameter ambiguity in affine parameterization.

Normal coordinates

Definition. Let M be a manifold with a connection ∇ . Let $p \in M$. The *exponential map* from $T_p(M)$ to M is defined as the map which sends X_p to the point unit affine parameter distance along the geodesic through p with tangent X_p at p .

Remark. It can be shown that this map is one-to-one and onto *locally*, i.e., for X_p in a neighbourhood of the origin in $T_p(M)$.

Exercise. Let $0 \leq t \leq 1$. Show that the exponential map sends tX_p to the point affine parameter distance t along the geodesic through p with tangent X_p at p .

Definition. Let $\{e_\mu\}$ be a basis for $T_p(M)$. *Normal coordinates at p* are defined in a neighbourhood of p as follows. Pick q near p . Then the coordinates of q are X^μ where X^μ is the element of $T_p(M)$ that maps to q under the exponential map.

Lemma. $\Gamma_{(\nu\rho)}^\mu(p) = 0$ in normal coordinates at p . For a torsion-free connection, $\Gamma_{\nu\rho}^\mu(p) = 0$ in normal coordinates at p .

Proof. From the above exercise, it follows that affinely parameterized geodesics through p are given in normal coordinates by $X^\mu(t) = tX_p^\mu$. Hence the geodesic equation reduces to

$$\Gamma_{\nu\rho}^\mu(X(t))X_p^\nu X_p^\rho = 0. \quad (4.40)$$

Evaluating at $t = 0$ gives that $\Gamma_{\nu\rho}^\mu(p)X_p^\nu X_p^\rho = 0$. But X_p is arbitrary, so the first result follows. The second result follows using the fact that torsion-free implies $\Gamma_{[\nu\rho]}^\mu = 0$ in a coordinate chart.

Remark. The connection components away from p *will not* vanish in general.

Lemma. On a manifold with a metric, if the Levi-Civita connection is used to define normal coordinates at p then $g_{\mu\nu,\rho} = 0$ at p .

Proof. Apply the previous lemma. We then have, at p ,

$$0 = 2g_{\mu\sigma}\Gamma_{\nu\rho}^\sigma = g_{\mu\nu,\rho} + g_{\mu\rho,\nu} - g_{\nu\rho,\mu} \quad (4.41)$$

Now symmetrize on $\mu\nu$: the final two terms cancel and the result follows.

Remark. Again, we emphasize, this is valid *only at the point p* . At any point, we can introduce normal coordinates to make the first partial derivatives of the metric vanish at that point. They will not vanish away for that point.

Lemma. On a manifold with metric one can choose normal coordinates at p so that $g_{\mu\nu,\rho}(p) = 0$ and also $g_{\mu\nu}(p) = \eta_{\mu\nu}$ (Lorentzian case) or $g_{\mu\nu}(p) = \delta_{\mu\nu}$ (Riemannian case).

Proof. We've already shown $g_{\mu\nu,\rho}(p) = 0$. Consider $\partial/\partial X^1$. The integral curve through p of this vector field is $X^\mu(t) = (t, 0, 0, \dots, 0)$ (since $X^\mu = 0$ at p). But,

from the above, this is the same as the geodesic through p with tangent vector e_1 at p . It follows that $\partial/\partial X^1 = e_1$ at p (since both vectors are tangent to the curve at p). Similarly $\partial/\partial X^\mu = e_\mu$ at p . But the choice of basis $\{e_\mu\}$ was arbitrary. So we are free to choose $\{e_\mu\}$ to be an orthonormal basis. $\partial/\partial X^\mu$ then defines an orthonormal basis at p too.

In summary, on a Lorentzian (Riemannian) manifold, we can choose coordinates in the neighbourhood of any point p so that the components of the metric at p are the same as those of the Minkowski metric in inertial coordinates (Euclidean metric in Cartesian coordinates), and the first partial derivatives of the metric vanish at p .

Definition. In a Lorentzian manifold a *local inertial frame at p* is a set of normal coordinates at p with the above properties.

Thus the assumption that spacetime is a Lorentzian manifold leads to a precise mathematical definition of a local inertial frame.